# Viva Fukaya!

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## 1 Fukaya categories

*Fukaya categories* were the topic of the 2009 Talbot. The idea is to associate to a symplectic manifold $M$ (with some restrictions and extra data) an object $\mathrm{Fuk}(M)$ that we can think of as an $\mathrm{A}_\infty$-category [7; 8; 9; 10; 11; 13; 22; 26, Chapter I], differential graded (dg) category [5; 12; 16, §1.3.1; 17; 27], or $\mathbb{Z}$-linear stable $\infty$-category [16, Chapter 1; 18, §D.1]. That is, there should be a functor

$$\begin{pmatrix}\text{some category of}\\ \text{symplectic manifolds}\end{pmatrix} \xrightarrow{\mathrm{Fuk}(-;\mathbb{Z})} \begin{pmatrix}\text{stable } \infty\text{-categories}\\ \text{over } \mathbb{Z}\end{pmatrix} .$$

**1.1 Idea.** Just as we use algebra to study geometry in algebraic geometry, we should use the algebra of Fukaya categories to study topology.

**1.2 Comparison.** Some more familiar categorical invariants of geometric objects are the functors

$$\begin{aligned} \mathsf{Rings}^{\mathrm{op}} &\xrightarrow{\mathrm{Mod}} \begin{pmatrix}\text{stable } \infty\text{-categories}\\ \text{over } \mathbb{Z}\end{pmatrix} \\ R &\longmapsto \mathrm{Mod}(R) \end{aligned}$$

or

$$\begin{aligned} \mathsf{Schemes}^{\mathrm{op}} &\xrightarrow{\mathrm{D^bCoh}} \begin{pmatrix}\text{stable } \infty\text{-categories}\\ \text{over } \mathbb{Z}\end{pmatrix} \\ X &\longmapsto \mathrm{D^bCoh}(X) \end{aligned} .$$

**1.3 Recent Work** (ancestry: Cohen–Jones–Segal [3; 4])**.** For certain symplectic manifolds $M$, we can make sense of an $\mathbb{S}$-*linear* refinement of the Fukaya category $\mathrm{Fuk}(M;\mathbb{S})$. This is still conjectural, and $\mathrm{Fuk}(M;\mathbb{S})$ is not yet constructed. Some people working on this are: Abouzaid, Abouzaid–Blumberg, Abouzaid–Blumberg–Kragh, Large, and Lurie–Tanaka.

**1.4 Expectation.** *There is an equivalence*

$$\mathrm{Fuk}(\mathrm{T}^*\mathbb{R}^\infty;\mathbb{S}) \simeq \mathsf{Sp}^{\mathrm{fin}}$$

*between the* $\mathbb{S}$*-linear (wrapped) Fukaya category of* $\mathrm{T}^*\mathbb{R}^\infty$ *and the* $\infty$*-category of finite spectra.*

**1.5 Remark.** Here $T^*\mathbb{R}^\infty$ is the stabilization of the point as a symplectic manifold. In fact, the $\mathbb{S}$-linear wrapped Fukaya category of $T^*\mathbb{R}^n$ is the same for any $n$; they should all be the $\infty$-category of finite spectra.

Morally speaking, $\mathrm{Fuk}(-;\mathbb{S})$ should define an equivalence from a localization of the $\infty$-category of symplectic manifolds to the $\infty$-category of stable $\infty$-categories.

# 2 Reminder on Morse theory

We start with a brief review of Morse theory and its relation to homotopy theory. For details, consult [19, Part I].

**2.1 Reminder.** Let $Q$ be a manfiold and $f: Q \to \mathbb{R}$ a *Morse function*, i.e., a smooth function with no degenerate critical points. Let $c \in Q$ be a critical point. By the Morse Lemma, there exists a coordinate chart $\phi: \mathbb{R}^n \hookrightarrow Q$ such that $\phi(0) = c$ and when restricted to this coordinate chart, the Morse function $f$ has a simple form:

$$f\phi(x) = f(c) - \sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{n} x_j^2 .$$

The number $k$ is independent of the choice of the coordinate chart, and is called the *Morse index* of $f$ at the critical point $c$. See Figure 1 for some examples of Morse functions with critical points and their indices labeled.

One reason why a Morse function $f: Q \to \mathbb{R}$ is useful from the perspective of homotopy theory is because provides a way to give $Q$ a cell decomposition with $k$-cells in bijection with the critical points of $f$ of index $k$. This is explained by how the topology of the sublevel sets of the Morse function change when passing through a critical point.

**2.2 Notation.** Given a smooth function $f: Q \to \mathbb{R}$ and $t \in \mathbb{R}$, we write $Q_t \coloneqq f^{-1}(-\infty, t]$.

**2.3 Proposition** [19, Theorem 3.1]**.** *Let $f: Q \to \mathbb{R}$ be a smooth function, and $a, b \in \mathbb{R}$ with $a < b$. If $f^{-1}[a,b]$ is compact and there are no critical values between a and b, then:*

(1) *The manifolds $Q_a$ and $Q_b$ are diffeomorphic.*

(2) *The manifold $Q_b$ deformation retracts onto the submanifold $Q_a$.*

**2.4 Proposition** [19, Theorem 3.2]**.** *Let $f: Q \to \mathbb{R}$ be a smooth function and $c \in Q$ a nondegenerate critical point of $f$ of index $k$. Write $v \coloneqq f(c)$ and let $\varepsilon > 0$ be a number such that $f^{-1}[v-\varepsilon, v+\varepsilon]$ is compact and $c$ is the only critical point contained in $f^{-1}[v-\varepsilon, v+\varepsilon]$. Then the manifold $Q_{v+\varepsilon}$ is homotopy equivalent to a space obtained from $Q_{v-\varepsilon}$ by attaching a $k$-cell.*

Using Propositions 2.3 and 2.4 one can show:

**2.5 Proposition** [19, Theorem 3.5]**.** *Let $f: Q \to \mathbb{R}$ be a Morse function and assume that for each $t \in \mathbb{R}$, the manifold $Q_t$ is compact. Then $Q$ is homotopy equivalent to a CW complex with one $k$-cell for each critical point of $f$ of index $k$.*

Figure 2 illustrates how to use a Morse function to build a torus by cells.

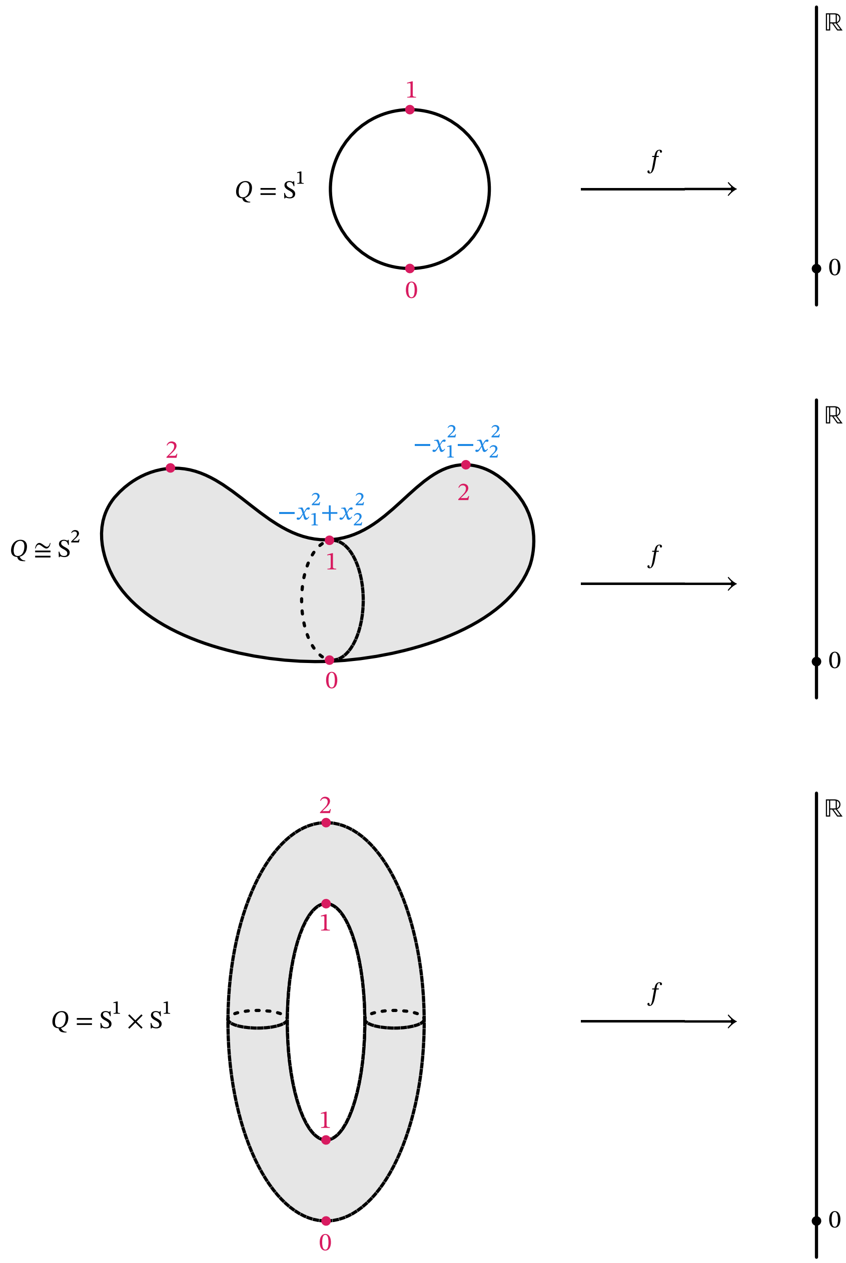


**Figure 1.** Three Morse functions. Each point is sent to its 'height' on the vertical $\mathbb{R}$ axis. Critical points are labeled in magenta with their Morse indices. The light blue labels are what the Morse function $f$ looks like near the critcal point.

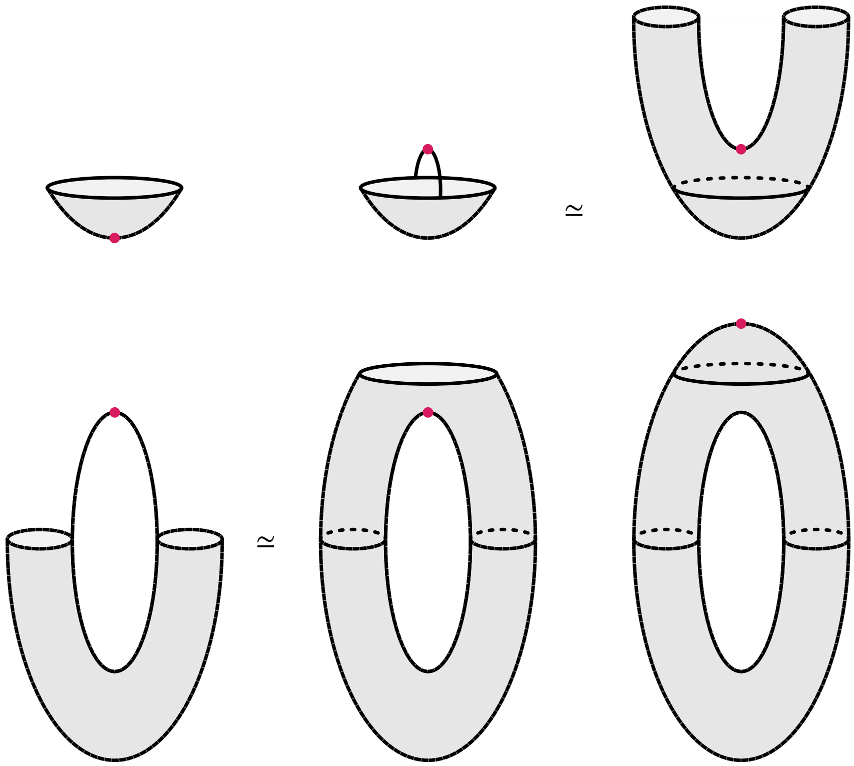

**Figure 2.** Using the height function from Figure 1 to give a cell decomposition of the torus.

## 3 Weinstein manifolds and sectors

Motivated by the relationship between Morse functions and CW complexes, we now give an indication of what a 'CW complex' should be in the context of symplectic geometry. We start by recalling the basics of symplectic manifolds.

**3.1 Recollection.** A *symplectic manifold* is the data of a pair $(M, \omega)$ of a $2n$-dimensional (smooth) manifold $M$ and a 2-form $\omega \in \Omega^2_{\mathrm{dR}}(M; \mathbb{R})$ satisfying:

(1) The $2n$-form $\omega^{\wedge n}$ is a *volume form* on $M$.

(2) The 2-form $\omega$ is *closed*: $\mathrm{d}\omega = 0$.

Note that (1) is equivalent to:

(1′) The map

$$\begin{aligned} \mathrm{T}M &\to \mathrm{T}^*M \\ v &\mapsto \omega(v, -) \end{aligned}$$

is an isomorphism of vector bundles over $M$.

The requirement that $\omega$ be closed constrains the topology of $M$:

**3.2 Observation.** If $(M, \omega)$ is a closed symplectic manifold of dimension $2n$, then Poincaré duality implies that the class $[\omega^{\wedge n}] \in \mathrm{H}^{2n}_{\mathrm{dR}}(M; \mathbb{R})$ is nonzero. Hence for each $1 \leq i \leq n$, the class $[\omega^{\wedge i}] \in \mathrm{H}^{2i}_{\mathrm{dR}}(M; \mathbb{R})$ is nonzero.

In particular, spheres other than $\mathrm{S}^0$ and $\mathrm{S}^2$ do not admit symplectic structures.

**3.3 Observation.** If $(M, \omega)$ is a symplectic manifold, then composition with the isomorphism $\mathrm{T}M \mathrel{\tilde{\to}} \mathrm{T}^*M$ of (1′) defines an isomorphism

$$\Gamma(M; \mathrm{T}M) \mathrel{\tilde{\to}} \Omega^1_{\mathrm{dR}}(M; \mathbb{R})$$

between vector fields on $M$ and differential 1-forms.

**3.4 Example.** Let $Q$ be an $n$-manifold. Then the cotangent bundle $\mathrm{T}^*Q$ has a naturally-defined symplectic form $\mathrm{d}p \wedge \mathrm{d}q$. The form $\mathrm{d}p \wedge \mathrm{d}q$ is defined locally on each coordinate patch of $Q$, so it is sufficient to explain the definition when $Q = \mathbb{R}^n$.

We write the coordinates of $\mathbb{R}^n$ as $(q_1, \dots, q_n)$ and the coordinates of

$$\mathrm{T}^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$$

in the cotangent direction as $(p_1, \dots, p_n)$. There is a naturally defined 1-form $\lambda$ on $\mathrm{T}^*\mathbb{R}^n$ given by the formula

$$\lambda \coloneqq \sum_{i=1}^{n} p_i \mathrm{d}q_i \ .$$

The symplectic form $\mathrm{d}p \wedge \mathrm{d}q$ is the differential

$$\mathrm{d}p \wedge \mathrm{d}q \coloneqq \mathrm{d}\lambda = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i \ .$$

The 1-form $\lambda$ is often called the *tautological* 1*-form* or *Liouville* 1*-form*.

Cotangent bundles are examples of a special type of symplectic manifold where the symplectic form is exact.

**3.5 Definition.** A symplectic manifold $(M, \omega)$ is *exact* if the symplectic form $\omega$ is exact, i.e., there exists a 1-form $\lambda$ such that $\omega = \mathrm{d}\lambda$. A 1-form whose differential is $\omega$ is called a *Liouville* 1*-form* or *primitive for* $\omega$. An *exact structure* on a symplectic manifold $(M, \omega)$ is a choice of a primitive $\lambda$ for $\omega$.

**3.6 Observation.** Let $(M, \omega)$ be a symplectic manifold. In light of Observation 3.3, given a 1-form $\lambda$ on $M$, there is a unique vector field $X$ such that $\iota_X \omega = \lambda$. Combining Cartan's homotopy formula for the Lie derivative

$$\mathcal{L}_X \omega = \mathrm{d}\iota_X \omega + \iota_X \mathrm{d}\omega$$

with the assumption that $\mathrm{d}\omega = 0$, we see that $\mathrm{d}\lambda = \omega$ if and only if $\mathcal{L}_X \omega = \omega$.

Thus the the data of a primitive for $\omega$ is equivalent to specifying a vector field $X$ satisfying $\mathcal{L}_X \omega = \omega$. Such a vector field is called a *Liouville vector field*.

**3.7 Remark.** In light of Observation 3.2, every positive-dimensional exact symplectic manifold is non-compact.

**3.8 Idea.** Given a Morse decomposition of a manifold $Q$, by applying $\mathrm{T}^*$ to the cells used to build up $Q$, we can build $\mathrm{T}^*Q$ symplectically. More generally, we should be able to make some symplectic manifolds out of 'Weinstein cells.'

For example, consider the height function on the circle from Figure 1. The Morse function tells us to build the circle out of two hemispheres as on the left-hand side of Figure 3. These hemispheres have vector fields generated by a choice of gradient flow, always flowing from lower index critical points to higher index critical points. It turns out one can construct another vector field on the cotangent bundles of these hemispheres by combining the original vector fields with an always-outward-pointing-in-the-cotangent-direction vector field. Since these vector fields agree at the boundaries of the hemispheres, we can glue the cotangent bundles together to obtain $\mathrm{T}^*\mathrm{S}^1$ by gluing together two cotangent bundles of Euclidean spaces.

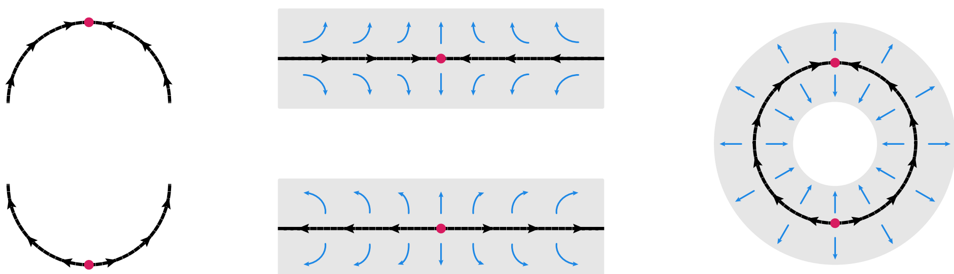

**Figure 3.** Gluing two copies of $\mathrm{T}^*\mathbb{R}$ together into $\mathrm{T}^*\mathrm{S}^1$.

We want to consider a larger class of symplectic manifolds that we can build out of cotangent bundles of Euclidean spaces as in Figure 3. Here is essentially what we need.

**3.9 Informal Definition.** A *Weinstein manifold* consists of:

(1) A symplectic manifold $(M, \omega)$.

(2) A Liouville vector field $X$ for $(M, \omega)$. Equivalently, a primitive $\lambda$ for $\omega$.

(3) A Morse function $f : M \to \mathbb{R}$.

Along with some additional properties of and compatibilities between the vector field $X$ and Morse function $f$.

A *Weinstein sector* is a generalization of a Weinstein manifold to have boundaries and corners.

**3.10 Slogan.** *Weinstein manifolds* are like CW complexes in the symplectic world.

See [2, Chapter 11; 6] for detailed introductions to Weinstein manifolds.

# 4 Lagrangians & 'wrapped' Fukaya categories

Now we explain (informally) how to define the Fukaya category $\mathrm{Fuk}(M)$. The objects of the Fukaya category are an important class of submanifold of a symplectic manifold:

**4.1 Definition.** Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. An $n$-dimensional submanifold $L \subset M$ is *Lagrangian* if $\omega|_L = 0$.

**4.2 Examples.**

(1) If $n = 1$ so that $M$ is a surface, then any curve in $M$ is Lagrangian. This is because $\omega$ is a 2-form, hence vanishes when restricted to any 1-dimensional submanifold.

(2) Let $Q$ be a manifold and write $\pi : \mathrm{T}^*Q \to Q$ for the projection. The zero section $Q \subset \mathrm{T}^*Q$ is Lagrangian. For a fixed $q \in Q$, the cotangent fiber $\mathrm{T}^*_qQ \coloneqq \pi^{-1}(q)$ is Lagrangian in $\mathrm{T}^*Q$. See Figure 4.

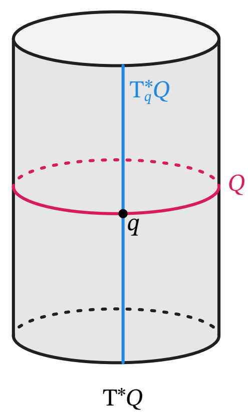


**Figure 4.** The cotangent bundle of a manifold $Q$ with the zero section in magenta and a cotangent fiber $\mathrm{T}^*_qQ$ in light blue.

In order to define $\mathrm{Fuk}(M)$, it turns out that we need to choose an *almost complex* structure compatible with the symplectic structure on $M$. Recall that an almost complex structure on $M$ is an endomorphism $J : \mathrm{T}M \to \mathrm{T}M$ of the tangent bundle of $M$ such that $J^2 = -\mathrm{id}_{\mathrm{T}M}$. Compatibility with a symplectic form $\omega \in \Omega^2_{\mathrm{dR}}(M;\mathbb{R})$ means that $\omega(-, J(-))$ defines a Riemannian metric on $M$.

Luckily, the space of compatible almost complex structures is contractible, so we shouldn't worry about this choice!

**4.3 Informal Definition** see [1; 26] for details**.** Let $(M, \omega)$ be a symplectic manifold equipped with a compatible almost complex structure $J : \mathrm{T}M \to \mathrm{T}M$. The $\mathbb{Z}/2$*-linear Fukaya category* $\mathrm{Fuk}(M;\mathbb{Z}/2)$ is an $\mathrm{A}_\infty$-category with:

(0) Objects: Lagrangian submanifolds $L \subset M$. (Literally speaking, this is *false*.)

(1) Suppose we are given Lagrangians $L_0, L_1 \subset M$. If $L_0$ and $L_1$ do not intersect transversely, then we deform $L_1$ so that the intersection $L_0 \cap L_1$ is transverse. Once $L_0$ and $L_1$ intersect transversely, we define the mapping chain complex $\mathrm{hom}(L_0, L_1)$ has underlying graded module

$$\mathrm{hom}(L_0, L_1) \coloneqq \bigoplus_{x \in L_0 \pitchfork L_1} \mathbb{Z}/2[|x|] .$$

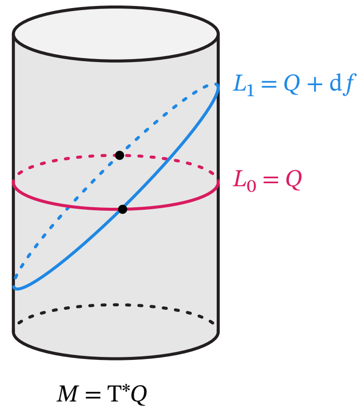


**Figure 5.** The cotangent bundle of a manifold $Q$ with the zero section $L_0$ and a deformation of the zero section so to make the self-intersection $L_0 \cap L_0$ transverse.

Here $|x|$ denotes the degree of the point $x$. The differential d on $\hom(L_0, L_1)$ is given by counting holomorphic disks:

$$\mathrm{d}(x) := \sum_y \# \left\{ \begin{matrix} \text{holomorphic} \\ \text{disks from} \\ x \text{ to } y \end{matrix} \right\} y\,.$$

By a *holomorphic disk from $x$ to $y$* we mean a map $u\colon\ \mathrm{D}^2 \to M$ as depicted in Figure 6 such that $\bar{\partial} u = 0$.

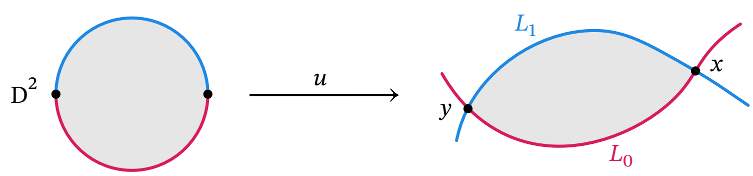


**Figure 6.** A holomorphic disk from $x$ to $y$.

(2) The $\mathrm{A}_\infty$-structure is defined heuristically as follows. Given Lagrangians $L_0, L_1, L_2 \subset M$, the composition

$$\hom(L_1, L_2) \otimes \hom(L_0, L_1) \to \hom(L_0, L_2)$$

is given by sending a pure tensor $y \otimes x$, where $y \in L_1 \pitchfork L_2$ and $x \in L_0 \pitchfork L_1$ to the sum

$$\sum_{z \in L_0 \pitchfork L_2} \# \left\{ \begin{matrix} \text{holomorphic} \\ \text{triangles with} \\ \text{vertices } x, y, \text{ and } z \end{matrix} \right\} z\,.$$

By a *holomorphic triangle with vertices $x$, $y$, and $z$* we mean a map $u\colon\ \mathrm{D}^2 \to M$ as depicted in Figure 7 such that $\bar{\partial} u = 0$. This composition is associative up to higher homotopies, given by counting holomorphic $k$-gons.

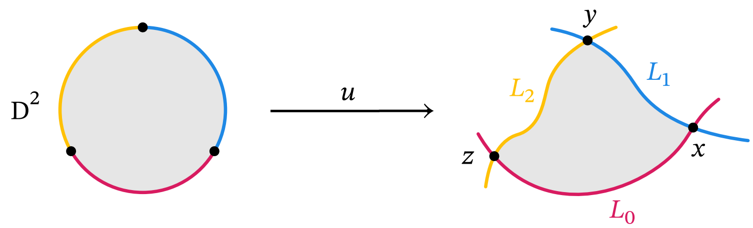


**Figure 7.** A holomorphic triangle with vertices $x$, $y$, and $z$.

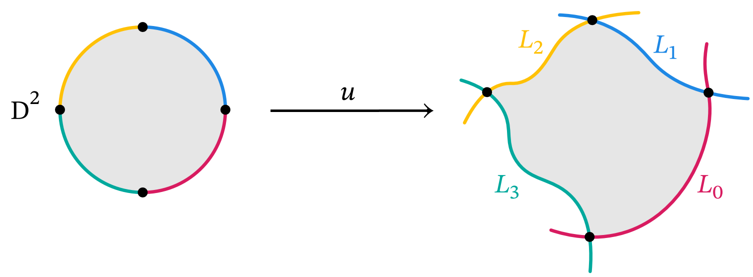


**Figure 8.** A holomorphic 4-gon.

# 5 Theorems & Conjectures

**5.1 Notation.** Given a Weinstein sector $M$, we write $\lambda_M$ for its chosen Liouville 1-form.

**5.2 Definition.** The category $\mathsf{Wein}$ of *Weinstein sectors* has objects Weinstein sectors and morphisms codimension 0 embeddings $j\colon M \hookrightarrow N$ such that

$$j^*\lambda_N = \lambda_M + \mathrm{d}\left(\begin{smallmatrix}\text{compactly supported}\\ \text{function}\end{smallmatrix}\right) .$$

**5.3 Theorem** (Oh–Tanaka [23; 24; 25])**.** *Let $M$ be a Weinstein sector. Then automorphisms of $M$ act on the $\mathbb{Z}$-linear stable $\infty$-category* $\mathrm{Fuk}(M;\mathbb{Z})$*. That is, the (wrapped) Fukaya category is functorial in automorphisms of Weinstein sectors.*

There is 'dimensional stabilization' $\mathsf{Wein}^{\diamond}$ of the category $\mathsf{Wein}$ informally described by taking the colimit of the diagram

$$\mathsf{Wein} \xrightarrow{(-)\times \mathrm{T}^*\mathbb{R}} \mathsf{Wein} \xrightarrow{(-)\times \mathrm{T}^*\mathbb{R}} \mathsf{Wein} \xrightarrow{(-)\times \mathrm{T}^*\mathbb{R}} \cdots$$

given by iterating the endofunctor $M \mapsto M \times \mathrm{T}^*\mathbb{R}$. The Fukaya category $\mathrm{Fuk}(-;\mathbb{Z})$ extends to $\mathsf{Wein}^{\diamond}$:

**5.4 Theorem** (Lazarev–Sylvan–Tanaka [15, Theorem 1.13])**.** *The (wrapped) Fukaya category defines a functor*

$$\mathrm{Fuk}(-;\mathbb{Z})\colon \mathsf{Wein}^{\diamond} \to \mathsf{StabCat}_{\infty,\mathbb{Z}}$$

*to $\mathbb{Z}$-linear stable $\infty$-categories.*

**5.5 Note.** There are ways to soup this up to ($\mathbb{S}$-linear) stable $\infty$-categories:

(1) The work of Abouzaid, Abouzaid–Blumberg, Abouzaid–Blumberg–Kragh, and Large mentioned in Recent Work 1.3.

(2) Lurie–Tanaka: Reformulate $\mathrm{Fuk}(M)$ as a solution to a deformation problem.

(3) Nadler–Tanaka [21]: Out of $M$ construct a stable $\infty$-category $\mathrm{Lag}(M)$ of *Lagrangian cobordisms*.

(4) Nadler–Shende [20]: microlocal spectral sheaves.

**5.6 Conjecture.** *The $\mathbb{S}$-linear (wrapped) Fukaya category defines a functor*

$$\mathrm{Fuk}(-;\mathbb{S})\colon \mathsf{Wein}^{\diamond} \to \mathsf{StabCat}_{\infty}$$

*to stable $\infty$-categories.*

**5.7 Conjecture** (see Nadler–Tanaka [21, Conjecture 1.6.1])**.** *There is an equivalence of $\mathbb{Z}$-linear stable $\infty$-categories*

$$\mathrm{Lag}(M) \otimes_{\mathbb{S}} \mathbb{Z} \simeq \mathrm{Fuk}(M;\mathbb{Z}) .$$

**5.8 Question.** Can the category $\mathsf{Wein}^{\diamond}$ be used to symplectically construct certain $\mathbb{E}_{\infty}$-rings? For example, $\mathbb{S}$ or $\mathbb{S}[1/p]$?

The answer is *yes* if we localize $\mathsf{Wein}^{\diamond}$ in a natural way!

**5.9 Theorem** (Lazarev–Sylvan–Tanaka [14])**.** *There is a (naturally defined) class of morphisms $W$ in $\mathsf{Wein}^{\diamond}$ that the functor $\mathrm{Fuk}(-;\mathbb{Z})$ carries to equivalences. Moreover, the following results hold:*

(1) *The localization $\mathsf{Wein}^{\diamond}[W^{-1}]$ has a natural symmetric monoidal structure.*

(2) *Given a prime number $p$, one can construct a symplectic manifold $D_p$ which is an $\mathbb{E}_{\infty}$-algebra in the symmetric monoidal $\infty$-category $\mathsf{Wein}^{\diamond}[W^{-1}]$.*

(3) *The Fukaya category $\mathrm{Fuk}(D_p;\mathbb{Z})$ has a natural symmetric monoidal structure and there is an equivalence of symmetric monoidal $\mathbb{Z}$-linear stable $\infty$-categories*

$$\mathrm{Fuk}(D_p;\mathbb{Z}) \simeq \mathrm{Mod}(\mathbb{Z}[1/p]) .$$

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