Closure properties of classes of maps

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0 Introduction

In this note, we prove the following result, as well as a generalization to n-localic coherent ∞-topoi:

0.1 Proposition (Example 2.36)**.** *Let $X \in \mathrm{Pro}(\mathbf{An}_\pi)$ be a profinite anima. Then there exists a profinite set Y and an effective epimorphism $Y \twoheadrightarrow X$ in $\mathrm{Pro}(\mathbf{An}_\pi)$.*

We claimed this result in [4, Proposition 13.4.9; 5, Proposition 3.3.8], however the proof there is incorrect. So the main purpose of this note is to correct the proof.

Since every π-finite anima admits an effective epimorphism from a finite set, Proposition 0.1 follows from the following more general result. To state it, let $\mathcal{C}$ be an ∞-category with finite limits, and let $\mathcal{P}$ be a class of maps in $\mathcal{C}$ that is closed under composition, contains all equivalences, and is stable under pullback.

0.2 Proposition (Proposition 2.22)**.** *Let $\mathcal{B} \subset \mathcal{C}$ be a full subcategory with the property that for each $X \in \mathcal{C}$, there exists a map $Y \to X$ in $\mathcal{P}$ with $Y \in \mathcal{B}$. Then for each $X \in \mathrm{Pro}(\mathcal{C})$, there exists a map $Y \to X$ in $\mathrm{Pro}(\mathcal{P})$ with $Y \in \mathrm{Pro}(\mathcal{B})$.*

0.3 Remark. Proposition 0.1 also follows from a more refined result of Lurie [SAG, Lemma E.1.6.6]. Lurie's proof makes use of model categories and uses constructions specific to working with simplicial sets. However, Lurie's proof inspired our proof of Proposition 0.2.

We prove Proposition 0.2 by making a number of choices to construct Y. A key point is that the pro-object X can always be represented by a diagram $X : S^{\mathrm{op}} \to \mathcal{C}$ where S is a filtered poset with the additional property that for each $s \in S$, the set $S_{\leq s} := \{ t \in S \mid t \leq s \}$ is finite (see Lemma 2.24). There is a *height function* $S \to \mathbf{N}$ sending s to the Krull dimension of $S_{\leq s}$; this height function provides a $\mathbf{N}$-indexed filtration on S. Using this filtration, one can try to construct Y inductively, starting with the height 0 elements (which are the minimal elements of S). In order for the inductive step to work, one needs the following closure property for the class $\mathcal{P}$, which may be of independent interest:

0.4 Proposition (Proposition 2.14)**.** *Let S be a finite poset and let $f: Y \to X$ be a map in* $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{C})$ *such that for each $s \in S$, the induced map*

$$Y_s \to X_s \underset{\lim_{t<s} X_t}{\times} \lim_{t<s} Y_s \tag{0.5}$$

is in $\mathcal{P}$. Then the induced map on limits $\lim_{s \in S^{\mathrm{op}}} Y_s \to \lim_{s \in S^{\mathrm{op}}} X_s$ *is in $\mathcal{P}$.*

0.6 Remark (Reedy fibrations)**.** Those familiar with Reedy categories may recognize the hypotheses of Proposition 0.4. The object $\lim_{t<s} X_s$ is the *matching object* of X at $s \in S$. The condition that (0.5) be in $\mathcal{P}$ is a version of the notion of a Reedy fibration in this context.

Proposition 0.4 seems to be a nontrivial closure property of classes of maps stable under pullback and composition. Our proof relies on an inductive description of (co)limits of diagrams indexed by posets of finite Krull dimension (Proposition 2.9 and Corollary 2.13), which may also be of independent interest.

We note that in the most simple case, Proposition 0.4 says that given a commutative diagram

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d, "f_1"'] & Y_0 \arrow[d, "f_0"] & Y_2 \arrow[l] \arrow[d, "f_2"] \\ X_1 \arrow[r] & X_0 & X_2 \arrow[l] \end{tikzcd},$$

if f_0 and the induced maps $Y_1 \to X_1 \times_{X_0} Y_0$ and $Y_2 \to X_2 \times_{X_0} Y_0$ are in $\mathcal{P}$, then the induced map on pullbacks $Y_1 \times_{Y_0} Y_2 \to X_1 \times_{X_0} X_2$ is in $\mathcal{P}$.

0.1 Linear overview

The main technical ingredient of this note (Proposition 0.4) is elementary in the sense that it only relies on more basic closure properties of a class $\mathcal{P}$ of maps stable under composition and pullback, together with a formula for limits over finite posets. As a result, aside from some specific applications and examples, we realized that this note could be completely self-contained, possibly at the cost of adding some length. Thus, in order to make this note accessible to a wider audience, we have decided to make it self-contained.

With that in mind, even though they are well-known to experts, in §1, we explain the more basic closure properties of the class $\mathcal{P}$ that we need. In §2, we give a formula for limits over posets of finite Krull dimension (Proposition 2.9 and Corollary 2.13), then prove Propositions 0.1, 0.2, and 0.4.

0.2 Notational conventions

We write **An** for the ∞-category of *anima* (also referred to as spaces or ∞-groupoids) and $\mathbf{Cat}_\infty$ for the ∞-category of ∞-categories.

0.3 Acknowledgments

We are very grateful to Tim Holzschuh, Marcin Lara, Catrin Mair, and Sebastian Wolf for noticing the error in the proofs given in [4, Proposition 13.4.9; 5, Proposition 3.3.8]. We thank Ben Antieau for helpful conversations on the contents of this note as well as for comments on a draft. In particular, we thank Ben for suggesting that we think beyond topos theory and include applications to n-complicial regular ∞-categories.

1 Basic closure properties

1.1 Reasonable classes of maps

We start by fixing some convenient terminology. First recall that given an ∞-category $\mathcal{C}$, a *class of maps* in $\mathcal{C}$ is a full subcategory $\mathcal{P} \subset \mathrm{Fun}([1], \mathcal{C})$ of the arrow ∞-category of $\mathcal{C}$.

1.1 Definition. Let $\mathcal{C}$ be an ∞-category with pullbacks. We say that a class of maps $\mathcal{P}$ in $\mathcal{C}$ is *reasonable* if the following conditions are satisfied:

(1) The class $\mathcal{P}$ contains all equivalences.

(2) The class $\mathcal{P}$ is closed under composition.

(3) The class $\mathcal{P}$ is stable under pullback.

There are countless examples of reasonable classes of maps. Here are some that we find particularly interesting or useful.

1.2 Example (properties of maps of schemes)**.** If $\mathcal{P}$ is the class of maps of schemes satisfying any of the following properties, then $\mathcal{P}$ is reasonable: affine, closed immersion, étale, finite, (locally) of finite type, (locally) of finite presentation, (faithfully) flat, integral, open immersion, proper, quasicompact, quasiseparated, qcqs, separated, universally closed.

1.3 Example (regular ∞-categories)**.** An ∞-category $\mathcal{C}$ with finite limits is *regular* if for every map $f : Y \to X$ in $\mathcal{C}$, the Čech nerve $Y^{\times_X \bullet}$ of f admits a geometric realization, and geometric realizations of Čech nerves are universal [11, Definition 2.1.1]. Examples of regular ∞-categories include local ∞-pretopoi [SAG, Definition A.6.1.1] and ∞-topoi. In particular, in a regular ∞-category, the class of effective epimorphisms is a reasonable class of maps.

1.4 Example (Kan fibrations)**.** Let $\mathcal{C}$ be a regular ∞-category. Then using the notion of an effective epimorphism, one can define the notions of Kan fibrations and trivial Kan fibrations of (semi)simplicial objects in $\mathcal{C}$. See [SAG, §A.5.2; 11, §4.1]. Both Kan fibrations and trivial Kan fibrations form reasonable classes of maps of (semi)simplicial objects.

1.5 Example (connectivity structures)**.** Let $\mathcal{C}$ be an ∞-category equipped with a *connectivity structure* in the sense of [6, Definition 2.1]. Then for each $n \in \mathbf{Z}$, the class of n-connected maps in $\mathcal{C}$ is reasonable.

1.6 Example (factorization systems)**.** Let $\mathcal{C}$ be an ∞-category with finite limits and let $(\mathcal{L}, \mathcal{R})$ be a factorization system on $\mathcal{C}$, so that every map f in $\mathcal{C}$ admits a factorization $f = r\ell$ where $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$. Then $\mathcal{R}$ is a reasonable class of maps [2, Lemma 3.1.6].

The following two classes of examples from the theory of ∞-topoi are due to Anel–Biedermann–Finster–Joyal [1; 2].

1.7 Example (modalities on ∞-topoi)**.** Let $\mathcal{X}$ be an ∞-topos. A *modality* on $\mathcal{X}$ is a factorization system $(\mathcal{L}, \mathcal{R})$ in which $\mathcal{L}$ is stable under pullback (see [2, Definition 3.4.1]). In this case, $\mathcal{L}$ is also a reasonable class of maps. Here are some examples of modalities:

(1) For an integer $n \geq -2$, the classes $\mathcal{L}$ of n-connected maps in $\mathcal{X}$ and $\mathcal{R}$ of n-truncated maps in $\mathcal{L}$ form a modality on $\mathcal{X}$.

(2) Let $F : \mathcal{X} \to \mathcal{Y}$ is a left exact localization, and let $\mathcal{L}$ be the class of F-equivalences and $\mathcal{R}$ the class of F-local maps. Then $(\mathcal{L}, \mathcal{R})$ forms a modality on $\mathcal{X}$ [1, Lemma 2.6.4].

1.8 Example (Goodwillie calculus)**.** Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{C}$ be a small ∞-category with finite colimits and a terminal object and let $n \in \mathbf{N}$. Then the inclusion $\mathrm{Exc}^n(\mathcal{C}, \mathcal{X}) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{X})$ of the full subcategory of n-excisive functors admits a left exact accessible left adjoint P_n. See [HA, Theorem 6.1.1.10 & Remark 6.1.1.11; 1, §3.1]. In particular, the P_n-equivalences and P_n-local maps form a modality on the ∞-topos $\mathrm{Fun}(\mathcal{C}, \mathcal{X})$.

1.2 Basic closure properties

Now we turn to proving the most basic closure properties of reasonable classes of maps. First recall that since limits commute, we have:

1.9 Lemma. *Let $\mathcal{C}$ be an ∞-category with pullbacks and*

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Z_1 \arrow[d] & Y_1 \arrow[l] \arrow[d] \\ X_0 \arrow[r] & Z_0 & Y_0 \arrow[l] \\ X_2 \arrow[r] \arrow[u] & Z_2 \arrow[u] & Y_2 \arrow[l] \arrow[u] \end{tikzcd} \tag{1.10}$$

a commutative diagram in $\mathcal{C}$. Then the limit of the diagram (1.10) *exists and is equivalent to both of the following two iterated pullbacks:*

(1) *Form the pullback of the columns of* (1.10)*, then take the pullback of the resulting cospan*

$$X_1 \times_{X_0} X_2 \longrightarrow Z_1 \times_{Z_0} Z_2 \longleftarrow Y_1 \times_{Y_0} Y_2 .$$

(2) *Form the pullback of the rows of* (1.10)*, then take the pullback of the resulting cospan*

$$X_1 \times_{Z_1} Y_1 \longrightarrow X_0 \times_{Z_0} Y_0 \longleftarrow X_2 \times_{Z_2} Y_2 .$$

The following consequence of Lemma 1.9 is quite useful in proving our first closure property.

1.11 Example. Let $\mathcal{C}$ be an ∞-category with pullbacks and let $f\colon X \to Z$, $g\colon Y \to Z$, and $z\colon Z \to Z'$ be maps in $\mathcal{C}$. Then the natural square

$$\begin{tikzcd}
X \times_Z Y \arrow[r, "\mathrm{id}_X \times_z \mathrm{id}_Y"] \arrow[d] & X \times_{Z'} Y \arrow[d, "f \times_{Z'} g"] \\
Z \arrow[r, "\Delta_z"'] & Z \times_{Z'} Z
\end{tikzcd} \tag{1.12}$$

is a pullback. To see this, apply Lemma 1.9 to the diagram

$$\begin{tikzcd}
X \arrow[r, "zf"] \arrow[d, "f"'] & Z' \arrow[d, equal] & Y \arrow[l, "zg"'] \arrow[d, "g"] \\
Z \arrow[r, "z"] \arrow[d, equal] & Z' & Z \arrow[l, "z"'] \arrow[d, equal] \\
Z \arrow[r, equal] & Z \arrow[u, "z"] \arrow[r, equal] & Z .
\end{tikzcd}$$

Taking pullbacks vertically then horizontally yields $X \times_Z Y$. On the other hand, taking pullbacks horizontally yields the cospan in (1.12) given by the removing the initial vertex of the square.

We now arrive at our first closure property:

1.13 Lemma. *Let $\mathcal{C}$ be an ∞-category with pullbacks and let $\mathcal{P}$ be a reasonable class of maps in $\mathcal{C}$. Consider a commutative diagram*

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d, "f_1"'] & Y_0 \arrow[d, "f_0"] & Y_2 \arrow[l] \arrow[d, "f_2"] \\
X_1 \arrow[r] & X_0 & X_2 \arrow[l]
\end{tikzcd} \tag{1.14}$$

in $\mathcal{C}$. Assume that one of the following conditions holds:

(1) *The induced map $Y_1 \to X_1 \times_{X_0} Y_0$ is in $\mathcal{P}$ (e.g., the left-hand square is a pullback) and f_2 is in $\mathcal{P}$.*

(2) *The maps f_1, f_2, and diagonal $\Delta_{f_0}\colon Y_0 \to Y_0 \times_{X_0} Y_0$ are in $\mathcal{P}$.*

(3) *The induced maps $Y_1 \to X_1 \times_{X_0} Y_0$ and $Y_2 \to X_2 \times_{X_0} Y_0$ are in $\mathcal{P}$ and f_0 is in $\mathcal{P}$.*

Then the induced map on pullbacks $Y_1 \times_{Y_0} Y_2 \to X_1 \times_{X_0} X_2$ is in $\mathcal{P}$.

1.15 Warning. One might hope that if f_0, f_1, and f_2 are all in $\mathcal{P}$, then the induced map on pullbacks $Y_1 \times_{Y_0} Y_2 \to X_1 \times_{X_0} X_2$ is in $\mathcal{P}$. However, this is not generally true: it is false even when $\mathcal{C}$ is the category of sets and $\mathcal{P}$ is the class of surjections.

Before proving Lemma 1.13, we give an example.

1.16 Example. Let $\mathcal{X}$ be an ∞-topos and let $n \geq -2$ be an integer. Consider a commutative diagram (1.14) where f_1 and f_2 are n-connected and f_0 is $(n+1)$-connected. Then, by definition, Δ_{f_0} is n-connected. Hence Lemma 1.13 shows that the induced map $Y_1 \times_{Y_0} Y_2 \to X_1 \times_{X_0} X_2$ is in n-connected. This recovers [8, Proposition 4.13].

Proof of Lemma 1.13. For (1), we first prove the claim under the stronger assumption that the left-hand square is a pullback. Consider the commutative cube

$$
\begin{tikzcd}
Y_1 \times_{Y_0} Y_2 \arrow[rr] \arrow[dr] \arrow[dd] & & Y_2 \arrow[dr, "f_2"] \arrow[dd] & \\
& X_1 \times_{X_0} X_2 \arrow[rr, crossing over] & & X_2 \arrow[dd] \\
Y_1 \arrow[rr] \arrow[dr, "f_1"'] & & Y_0 \arrow[dr, "f_0"] & \\
& X_1 \arrow[rr] \arrow[from=uu, crossing over] & & X_0 .
\end{tikzcd}
\tag{1.17}
$$

By definition, the front and back vertical faces are pullback squares. By assumption, the bottom horizontal face is also a pullback square. By the gluing lemma for pullbacks, the top horizontal face is also a pullback. Since $f_2 \in \mathcal{P}$ and the class $\mathcal{P}$ is stable under pullback, we deduce that the induced map on pullbacks $Y_1 \times_{Y_0} Y_2 \to X_1 \times_{X_0} X_2$ is also in $\mathcal{P}$.

Now we treat the general case of (1), only assuming that $Y_1 \to X_1 \times_{X_0} Y_0$ and f_2 are in $\mathcal{P}$. For this, consider the commutative diagram

$$
\begin{tikzcd}
Y_1 \arrow[r] \arrow[d] & Y_0 \arrow[d, equal] & Y_2 \arrow[l] \arrow[d, equal] \arrow[dl, phantom, "\llcorner", very near start] \\
X_1 \times_{X_0} Y_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 \arrow[d, "f_0"] & Y_2 \arrow[l] \arrow[d, "f_2"] \\
X_1 \arrow[r] & X_0 & X_2 . \arrow[l]
\end{tikzcd}
$$

By the special case, the map from the pullback of the top horizontal span to the pullback of the middle horizontal span is in $\mathcal{P}$. Again by the special case, the map from the pullback of the middle horizontal span to the pullback of the bottom horizontal span is in $\mathcal{P}$. Since $\mathcal{P}$ is closed under composition, we deduce that the the induced map on pullbacks $Y_1 \times_{Y_0} Y_2 \to X_1 \times_{X_0} X_2$ is in $\mathcal{P}$.

Now we prove (2). First note that the natural map $Y_1 \times_{Y_0} Y_2 \to X_1 \times_{X_0} X_2$ factors as a composite

$$
Y_1 \times_{Y_0} Y_2 \longrightarrow Y_1 \times_{X_0} Y_2 \longrightarrow X_1 \times_{X_0} X_2 .
$$

Since $\mathcal{P}$ is stable under composition, it suffices to prove that each of these maps is in $\mathcal{P}$. (That is, we need to show the claim in the special cases when f_1 and f_2 are identities, and when f_0 is the identity.)

For the right-hand map, note that the map $Y_1 \times_{X_0} Y_2 \to X_1 \times_{X_0} X_2$ factors as a composite

$$Y_1 \times_{X_0} Y_2 \xrightarrow{f_1 \times_{X_0} \mathrm{id}_{Y_2}} X_1 \times_{X_0} Y_2 \xrightarrow{\mathrm{id}_{X_1} \times_{X_0} f_2} X_1 \times_{X_0} X_2 .$$

Since $f_1, f_2 \in \mathcal{P}$ and $\mathcal{P}$ is stable under pullback, both of the above maps are in $\mathcal{P}$; hence so is their composite

That the left-hand map is in $\mathcal{P}$ follows from the fact that the natural square

$$\begin{tikzcd}
Y_1 \times_{Y_0} Y_2 \arrow[r, "\mathrm{id}_{Y_1} \times_{f_0} \mathrm{id}_{Y_2}"] \arrow[d] & Y_1 \times_{X_0} Y_2 \arrow[d, "f_1 \times_{X_0} f_2"] \\
Y_0 \arrow[r, "\Delta_{f_0}"'] & Y_0 \times_{X_0} Y_0
\end{tikzcd}$$

is a pullback (Example 1.11), the assumption that $\Delta_{f_0} \in \mathcal{P}$, and the assumption that $\mathcal{P}$ is stable under pullback.

To prove (3), consider the commutative diagram

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d] & Y_0 \arrow[d, equal] & Y_2 \arrow[l] \arrow[d] \\
X_1 \times_{X_0} Y_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y_0 \arrow[d, "f_0"] & X_2 \times_{X_0} Y_0 \arrow[l] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \\
X_1 \arrow[r] & X_0 & X_2 \arrow[l]
\end{tikzcd}$$

Since the induced maps $Y_1 \to X_1 \times_{X_0} Y_0$ and $Y_2 \to X_2 \times_{X_0} Y_0$ are in $\mathcal{P}$, by part (2), the map from the pullback of the top horizontal cospan to the pullback of the middle horizontal cospan is in $\mathcal{P}$. Since $f_0 \in \mathcal{P}$ and $\mathcal{P}$ is stable under pullback, all of the vertical maps in the bottom half of the diagram are in $\mathcal{P}$. Hence by part (1), the map from the pullback of the middle horizontal cospan to the pullback of the bottom horizontal cospan is in $\mathcal{P}$. Since $\mathcal{P}$ is closed under composition, we deduce that the natural map $Y_1 \times_{Y_0} Y_2 \to X_1 \times_{X_0} X_2$ is in $\mathcal{P}$. □

An important consequence of Lemma 1.13 is that reasonable classes of maps are closed under finite products.

1.18 Corollary. *Let $\mathcal{C}$ be an ∞-category with finite limits and let $\mathcal{P}$ be a reasonable class of maps in $\mathcal{C}$. Then given a finite set $(f_i \colon Y_i \to X_i)_{i \in I}$ of maps in $\mathcal{P}$, the product*

$$\prod_{i \in I} f_i \colon \prod_{i \in I} Y_i \to \prod_{i \in I} X_i$$

is also in $\mathcal{P}$.

Proof. It suffices to treat the case $\#I = 2$. For this, apply Lemma 1.13 to the diagram

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d, "f_1"'] & 1_{\mathcal{C}} \arrow[d, equal] & Y_2 \arrow[l] \arrow[d, "f_2"] \\
X_1 \arrow[r] & 1_{\mathcal{C}} & X_2 \arrow[l]
\end{tikzcd}$$

□

1.3 Cancellation

We conclude this section by proving a cancellation result for reasonable classes of maps. This is not needed in §2, but is quite useful. Especially in algebraic geometry.

1.19 Recollection. Let $\mathcal{C}$ be an ∞-category with finite products and $f : X \to Y$ a map in $\mathcal{C}$. The *graph* of f is the map $\mathrm{gr}_f : X \to X \times Y$ induced by the universal property of the product by the identity $X \to X$ and $f : X \to Y$.

1.20. If $\mathcal{C}$ has pullbacks, $Z \in \mathcal{C}$, and $f : X \to Y$ is a map in $\mathcal{C}_{/Z}$, then note that the graph of f in $\mathcal{C}_{/Z}$ is a map $X \to X \times_Z Y$.

1.21 Lemma. *Let $\mathcal{C}$ be an ∞-category with pullbacks and let $f : X \to Y$ and $g : Y \to Z$ be maps in $\mathcal{C}$. Then the square*

$$\begin{tikzcd}
X \arrow[r, "\mathrm{gr}_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d, "f \times_Z \mathrm{id}_Y"] \\
Y \arrow[r, "\Delta_g"'] & Y \times_Z Y
\end{tikzcd}$$

is a pullback square.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\mathrm{gr}_f"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_X"] & X \times_Z Y \arrow[r, "\mathrm{pr}_1"'] \arrow[d, "f \times_Z \mathrm{id}_Y"] & X \arrow[d, "f"] \\
Y \arrow[r, "\Delta_g"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Y \times_Z Y \arrow[r, "\mathrm{pr}_1"] & Y .
\end{tikzcd}$$

By cancellation for pullbacks, the right-hand square is a pullback. The large outer rectangle is clearly a pullback square. By the pasting lemma for pullback squares, the left-hand square is also a pullback, as desired. □

The following is the promised cancellation result.

1.22 Lemma (cancellation)**.** *Let $\mathcal{C}$ be an ∞-category with pullbacks and let $\mathcal{P}$ be a reasonable class of maps in $\mathcal{C}$. Given maps $f : X \to Y$ and $g : Y \to Z$, if $gf \in \mathcal{P}$ and $\Delta_g \in \mathcal{P}$, then $f \in \mathcal{P}$.*

Proof. By Example 1.11 and Lemma 1.21 have pullback squares

$$\begin{tikzcd}
X \arrow[r, "\mathrm{gr}_f"] \arrow[d, "f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & X \times_Z Y \arrow[d, "f \times_Z \mathrm{id}_Y"] \\
Y \arrow[r, "\Delta_g"'] & Y \times_Z Y
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \times_Z Y \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "g"] \\
X \arrow[r, "gf"'] & Z .
\end{tikzcd}$$

Since $\Delta_g, gf \in \mathcal{P}$ and $\mathcal{P}$ is stable under pullback, we see that gr_f and $\mathrm{pr}_2 : X \times_Z Y \to Y$ are in $\mathcal{P}$. Since $\mathcal{P}$ is closed under composition, we deduce that $\mathrm{pr}_2 \circ \mathrm{gr}_f = f$ is in $\mathcal{P}$. □

2 Limits over posets & covering pro-objects

Let $\mathcal{C}$ be an ∞-category with finite limits, let $\mathcal{P}$ be a reasonable class of maps in $\mathcal{C}$, and let $\mathcal{B} \subset \mathcal{C}$ be a full subcategory with the property that for each $X \in \mathcal{C}$, there exists a map $Y \to X$ in $\mathcal{P}$ with $Y \in \mathcal{B}$. The main goal of this section is to show that for every pro-object $X \in \mathrm{Pro}(\mathcal{C})$, there exists a map $f : Y \to X$ in $\mathrm{Pro}(\mathcal{P})$ where $Y \in \mathrm{Pro}(\mathcal{B})$. See Proposition 2.22. The main example of such a situation is where $\mathcal{C} = \mathbf{An}_\pi$ is the ∞-category of π-finite anima, $\mathcal{P}$ is the class of effective epimorphisms, and $\mathcal{B} = \mathbf{Set}_{\mathrm{fin}}$ is the category of finite sets.

The key technical input is a closure property for $\mathcal{P}$ under limits indexed by finite posets, explained in Proposition 0.4 of the introduction. To establish it, we start in §2.1 by giving an inductive formula for limits over finite dimensional posets. In §2.2, we establish the desired closure property. Subsection 2.3 is dedicated to applying this closure property to cover pro-objects.

2.1 A formula for limits over finite dimensional posets

We begin by fixing some conventions for dimensions of posets and heights of elements of posets.

2.1 Notation. Let S be a poset and $s \in S$. We write

$$S_{\leq s} := \{ t \in S \mid t \leq s \} \qquad \text{and} \qquad S_{<s} := \{ t \in S \mid t < s \} .$$

We regard $S_{\leq s}$ and $S_{<s}$ as subposets of S.

2.2 Definition. Let S be a poset. The *(Krull) dimension* of S is the supremum

$$\dim(S) := \sup \{ n \in \mathbf{N} \mid \text{there is a chain of strict inequalities } s_0 < \cdots < s_n \text{ in } S \} .$$

Given an element $s \in S$, the *height* of s is the dimension

$$\mathrm{ht}(s) := \dim(S_{\leq s}) .$$

2.3. Note that $\mathrm{ht}(s) = 0$ if and only if s is a minimal element of S. Similarly, if S is a poset of dimension $n \in \mathbf{N}$, then $\mathrm{ht}(s) = n$ if and only if s is a maximal element of S. Also note that if s and t are two elements of the same finite height, then s and t are incomparable.

2.4 Remark. If A is a ring and $\mathrm{Spec}(A)$ is the set of prime ideals regarded as a poset under inclusion, then $\dim(\mathrm{Spec}(A))$ in the sense of Definition 2.2 recovers the Krull dimension of A. Moreover, for a prime $\mathfrak{p} \subset A$, the height of $\mathfrak{p}$ as an element of the poset $\mathrm{Spec}(A)$ agrees with the height of $\mathfrak{p}$ in the sense of commutative algebra.

2.5 Notation. Let S be a poset and $n \in \mathbf{N}$. We write

$$S_{\mathrm{ht} \leq n} := \{ s \in S \mid \mathrm{ht}(s) \leq n \} \qquad \text{and} \qquad S_{\mathrm{ht}=n} := \{ s \in S \mid \mathrm{ht}(s) = n \} .$$

We regard $S_{\mathrm{ht}\leq n}$ and $S_{\mathrm{ht}=n}$ as subposets of S.

We now proceed to give a formula for a (co)limit indexed by a poset of finite dimension. To do this, we construct an ∞-category out of $S_{\mathrm{ht}=n}$ and $S_{\mathrm{ht}\leq n-1}$ that approximates S.

2.6 Construction. Let $n \in \mathbf{N}$ and let S be a poset. We write $\mathrm{G}_n(S)$ for the ∞-category given by the pushout of the span

$$S_{\mathrm{ht}=n} \longleftarrow \coprod_{s \in S_{\mathrm{ht}=n}} S_{<s} \longrightarrow S_{\mathrm{ht}\leq n-1} .$$

Here, the left-hand map is induced by the universal property of the coproduct and the maps $S_{<s} \to \{s\} \hookrightarrow S_{\mathrm{ht}=n}$. The right-hand map is induced by the universal property of the coproduct and the inclusions $S_{<s} \hookrightarrow S_{\mathrm{ht}\leq n-1}$. Note that the ∞-category $\mathrm{G}_n(S)$ comes equipped with a natural functor $\phi \colon \mathrm{G}_n(S) \to S$.

2.7 Recollection. We write $\mathrm{B} \colon \mathbf{Cat}_\infty \to \mathbf{An}$ for the left adjoint to the inclusion $\mathbf{An} \hookrightarrow \mathbf{Cat}_\infty$. For an ∞-category $\mathcal{C}$, we call $\mathrm{B}\mathcal{C}$ the *classifying anima* of $\mathcal{C}$. We say that a functor $F \colon \mathcal{C} \to \mathcal{D}$ is a *weak homotopy equivalence* if $\mathrm{B}F$ is an equivalence. We say that ∞-category $\mathcal{C}$ is *weakly contractible* if $\mathrm{B}\mathcal{C} \simeq *$.

2.8 Recollection. We follow the terminology of Clausen and Jansen in [7, Theorem 2.19] and say that a functor of ∞-categories $f \colon \mathcal{I} \to \mathcal{J}$ is a colim-*equivalence* if for every ∞-category $\mathcal{C}$ and functor $X \colon \mathcal{J} \to \mathcal{C}$, the colimit $\operatorname{colim}_{j \in \mathcal{J}} X_j$ exists if and only if $\operatorname{colim}_{i \in \mathcal{I}} X_{f(i)}$ exists, and in this case the map

$$\operatorname*{colim}_{i \in \mathcal{I}} X_{f(i)} \to \operatorname*{colim}_{j \in \mathcal{J}} X_j$$

is an equivalence.[1] By Quillen's Theorem A [Ker, Tag 02NY], this holds if and only if for all $j \in \mathcal{J}$, the ∞-category $\mathcal{I} \times_{\mathcal{J}} \mathcal{J}_{j/}$ is weakly contractible.

Now for the main result of this subsection:

2.9 Proposition. *Let $n \in \mathbf{N}$ and let S be a poset of dimension n. Then the functor $\phi \colon \mathrm{G}_n(S) \to S$ is a* colim-*equivalence.*

2.10 Warning. The functor $\phi \colon \mathrm{G}_n(S) \to S$ is almost never an equivalence. For example, if S has a unique maximal element s, then $\mathrm{G}_n(S) \simeq \{s\}$ is the terminal poset.

Proof of Proposition 2.9. By Quillen's Theorem A, we need to check that for each $\sigma \in S$, the ∞-category $\mathrm{G}_n(S) \times_S S_{\geq\sigma}$ is weakly contractible. Note that the inclusion $S_{\geq\sigma} \hookrightarrow S$ is a left fibration (see [9, Lemma A.2.6]), in particular, an exponentiable fibration. Hence the functor $(-) \times_S S_{\geq\sigma}$ preserves colimits. Thus $\mathrm{G}_n(S) \times_S S_{\geq\sigma}$ is the pushout of the span

$$\coprod_{\substack{s \in S_{\mathrm{ht}=n} \\ s \geq \sigma}} \{s\} \longleftarrow \coprod_{\substack{s \in S_{\mathrm{ht}=n} \\ s \geq \sigma}} (S_{<s} \cap S_{\geq\sigma}) \longrightarrow S_{\mathrm{ht}\leq n-1} \cap S_{\geq\sigma} . \tag{2.11}$$

We separate the proof that $\mathrm{B}(\mathrm{G}_n(S) \times_S S_{\geq\sigma})$ is contractible into two cases.

Case 1: $\mathrm{ht}(\sigma) = n$. Equivalently, σ is a maximal element. In this case, the middle and right terms in the span (2.11) are empty, and the left term is $\{\sigma\}$. Since $\mathrm{B} \colon \mathbf{Cat}_\infty \to \mathbf{An}$ preserves colimits, we see that

$$\mathrm{B}(\mathrm{G}_n(S) \times_S S_{\geq\sigma}) \simeq \mathrm{B}\{\sigma\} = * .$$

Case 2: $\mathrm{ht}(\sigma) \leq n-1$. We first claim that the left-hand map in (2.11) is a weak homotopy equivalence. In fact, for each $s \in S_{\mathrm{ht}=n}$ with $s > \sigma$, the poset $S_{<s} \cap S_{\geq\sigma}$ is weakly contractible.

[1] Such functors are also commonly called *final* or *cofinal*, depending on the author.

This is because $S_{<s} \cap S_{\geq \sigma}$ has an initial object, namely σ. To conclude the proof, note that since B preserves colimits and $S_{\mathrm{ht} \leq n-1} \cap S_{\geq \sigma}$ also has initial object σ, we deduce that

$$\mathrm{B}(\mathrm{G}_n(S) \times_S S_{\geq \sigma}) \simeq \mathrm{B}(S_{\mathrm{ht} \leq n-1} \cap S_{\geq \sigma}) \simeq * \,. \qquad \square$$

2.12 Recollection. Let $\mathcal{C}$ be an ∞-category, let $\mathcal{I}_\bullet \colon \Lambda \to \mathbf{Cat}_\infty$ be a diagram of ∞-categories, and let $F \colon \operatorname{colim}_{\lambda \in \Lambda} \mathcal{I}_\lambda \to \mathcal{C}$ be a diagram in $\mathcal{C}$ indexed by the colimit of $\mathcal{I}_\bullet$. Then, provided all of the relevant limits exist, the limit

$$\lim_{i \in \operatorname{colim}_{\lambda \in \Lambda} \mathcal{I}_\lambda} F_i$$

is equivalent to the iterated limit $\lim_{\lambda \in \Lambda} \lim_{i \in \mathcal{I}_\lambda} F_i$.

2.13 Corollary. *Let $n \in \mathbf{N}$, let S be a (finite) poset of dimension n, and let $\mathcal{C}$ be an ∞-category with (finite) limits. Let $X \colon S^{\mathrm{op}} \to \mathcal{C}$ be a diagram. Then the natural square*

$$\begin{tikzcd}
\lim_{s \in S^{\mathrm{op}}} X_s \arrow[r] \arrow[d] & \lim_{t \in S^{\mathrm{op}}_{\mathrm{ht} \leq n-1}} X_t \arrow[d] \\
\prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} X_s \arrow[r] & \prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} \lim_{t<s} X_t
\end{tikzcd}$$

is a pullback square in $\mathcal{C}$.

Proof. By Proposition 2.9, the functor $\phi^{\mathrm{op}} \colon \mathrm{G}_n(S)^{\mathrm{op}} \to S^{\mathrm{op}}$ is a lim-equivalence. The claim now follows from the definition of $\mathrm{G}_n(S)$ as a pushout and Recollection 2.12. $\square$

2.2 Stability properties for limits over posets

We now prove the following closure property of reasonable classes of maps, which is the key technical result of this note.

2.14 Proposition. *Let $\mathcal{C}$ be an ∞-category with finite limits and let $\mathcal{P}$ be a reasonable class of maps in $\mathcal{C}$. Let S be a finite poset and let $f \colon Y \to X$ be a map in $\mathrm{Fun}(S^{\mathrm{op}}, \mathcal{C})$ such that for each $s \in S$, the induced map*

$$Y_s \to X_s \underset{\lim_{t<s} X_t}{\times} \lim_{t<s} Y_s \tag{2.15}$$

is in $\mathcal{P}$. Then the induced map on limits $\lim_{s \in S^{\mathrm{op}}} Y_s \to \lim_{s \in S^{\mathrm{op}}} X_s$ is in $\mathcal{P}$.

2.16 Remark. Note that if $s \in S$ is a minimal element, then the condition that the map (2.15) be in $\mathcal{P}$ says that the map $f_s \colon Y_s \to X_s$ is in $\mathcal{P}$.

2.17 Remark. In the special case where the poset S is $1 \leftarrow 0 \rightarrow 2$, Proposition 2.14 recovers Lemma 1.13-(3).

2.18 Remark. If the ∞-category $\mathcal{C}$ admits small limits and $\mathcal{P}$ is closed under small products, our proof shows that Proposition 2.14 is true under the weaker assumption that the poset S is finite dimensional.

Proof of Proposition 2.14. We prove the claim by induction on the dimension of S. In the base case $\dim(S) = 0$, the poset S is just a set. The claim then follows from the fact that $\mathcal{P}$ is closed under finite products (Corollary 1.18).

For the induction step, let $n \geq 1$, assume that $\dim(S) = n$ and that the claim has been proven for all finite posets of dimension $\leq n-1$. Consider the commutative diagram

$$\begin{tikzcd} \prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} Y_s \arrow[r] \arrow[d] & \prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} \lim_{t<s} Y_t \arrow[d] & \lim_{t \in S^{\mathrm{op}}_{\mathrm{ht}\leq n-1}} Y_t \arrow[l] \arrow[d] \\ \prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} X_s \arrow[r] & \prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} \lim_{t<s} X_t & \lim_{t \in S^{\mathrm{op}}_{\mathrm{ht}\leq n-1}} X_t \arrow[l] \end{tikzcd}$$

where the vertical maps are induced by the natural transformation $f\colon Y \to X$. Since we have $\dim(S_{\mathrm{ht}\leq n-1}) = n-1$, by the inductive hypothesis, the right-hand vertical map is in $\mathcal{P}$. Note that since limits commute, the induced map from $\prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} Y_s$ to the pullback of the left-hand cospan is the product of the induced maps

$$Y_s \to X_s \underset{\lim_{t<s} X_t}{\times} \lim_{t<s} Y_s \, .$$

By assumption, each of these maps is in $\mathcal{P}$. Since $\mathcal{P}$ is closed under finite products (Corollary 1.18), we deduce that the induced map from $\prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} Y_s$ to the pullback of the left-hand cospan is also in $\mathcal{P}$. By Lemma 1.13 and the description of limits over posets of finite dimension (Corollary 2.13), we conclude that the induced map

$$\lim_{s \in S^{\mathrm{op}}} Y_s \to \lim_{s \in S^{\mathrm{op}}} X_s$$

is also in $\mathcal{P}$. □

We conclude this subsection with an application of Corollary 2.13 and Proposition 2.14 to simplicial objects in a regular ∞-category (in the sense of Example 1.3). Since geometric realizations of simplicial objects do not generally commute with finite limits, this application is not obvious from the definitions.

2.19 Proposition. *Let $\mathcal{X}$ be a regular ∞-category, let S be a finite poset, and let*

$$Y\colon S^{\mathrm{op}} \to \mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{X})$$

be a diagram of simplicial objects. Assume that for each $s \in S$, the induced map $Y_s \to \lim_{t<s} Y_t$ is a Kan fibration. Then:

(1) *The limit $\lim_{s \in S^{\mathrm{op}}} Y_s$ satisfies the Kan condition.*

(2) *If $\mathcal{X}$ is a hypercomplete ∞-topos, then natural map $|\lim_{s \in S^{\mathrm{op}}} Y_s| \to \lim_{s \in S^{\mathrm{op}}} |Y_s|$ is an equivalence.*

Proof. We first prove (1). Recall from Example 1.4 that the class of Kan fibrations is reasonable. Let X denote the terminal object of $\mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{X})$, i.e., the constant diagram with value the terminal object of $\mathcal{X}$. Let $f : Y \to X$ denote the unique map. Since the limit of the constant diagram with value the terminal object is still the terminal object, our assumption on Y is equivalent to the condition that for each $s \in S$, the induced map

$$Y_s \to X_s \underset{\lim_{t<s} X_t}{\times} \lim_{t<s} Y_s$$

is a Kan fibration. Hence applying Proposition 2.14 to $f : Y \to X$, we see that the map

$$\lim_{s \in S^{\mathrm{op}}} Y_s \to \lim_{s \in S^{\mathrm{op}}} X_s \simeq 1_{\mathrm{Fun}(\boldsymbol{\Delta}^{\mathrm{op}}, \mathcal{X})}$$

is a Kan fibration. Since the target is the terminal object, this is equivalent to saying that $\lim_{s \in S^{\mathrm{op}}} Y_s$ satisfies the Kan condition.

We prove (2) by induction on the dimension of S. In the base case $\dim(S) = 0$, the poset S is just a finite set. In this case, since $\boldsymbol{\Delta}^{\mathrm{op}}$ is sifted and S is finite, we deduce that the natural map

$$\Big| \prod_{s \in S^{\mathrm{op}}} Y_s \Big| \to \prod_{s \in S^{\mathrm{op}}} |Y_s|$$

is an equivalence.

For the induction step, let $n \geq 1$, assume that $\dim(S) = n$ and that we have proven the claim for all finite posets of dimension $\leq n - 1$. By Corollary 2.13, the natural square

$$\begin{tikzcd} \lim_{s \in S^{\mathrm{op}}} Y_s \arrow[r] \arrow[d] & \lim_{t \in S^{\mathrm{op}}_{\mathrm{ht} \leq n-1}} Y_t \arrow[d] \\ \prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} Y_s \arrow[r] & \prod_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}} \lim_{t<s} Y_t \end{tikzcd} \tag{2.20}$$

is a pullback. Since $\mathcal{X}$ is a hypercomplete ∞-topos, geometric realization commutes with pullbacks along Kan fibrations [SAG, Theorem A.5.4.1]. Thus, again applying Corollary 2.13, it suffices to show that the bottom horizontal map in (2.20) is a Kan fibration and that geometric realization commutes with each of the limits in the three non-initial vertices of the square (2.20).

First, note that that the bottom horizontal map in (2.20) is a finite product of the natural maps $Y_s \to \lim_{t<s} Y_t$. By assumption, each of these maps is a Kan fibration. The closure of Kan fibrations under finite products implies that the bottom horizontal map is a Kan fibration.

Now we show that that geometric realization commutes with each of the limits in the three non-initial vertices of the square (2.20). For the lower left-hand corner, this follows from the fact that $\boldsymbol{\Delta}^{\mathrm{op}}$ is sifted. For the upper right-hand corner, note that since $\dim(S_{\mathrm{ht} \leq n-1}) = n - 1$, by the inductive hypothesis, the natural map

$$\Big| \lim_{t \in S^{\mathrm{op}}_{\mathrm{ht} \leq n-1}} Y_t \Big| \to \lim_{t \in S^{\mathrm{op}}_{\mathrm{ht} \leq n-1}} |Y_t|$$

is an equivalence. For the lower right-hand corner, note that since $\boldsymbol{\Delta}^{\mathrm{op}}$ is sifted, geometric realization commutes with the product appearing in that corner. Thus it suffices to show that for each $s \in S_{\mathrm{ht}=n}$, the natural map

$$\Big| \lim_{t<s} Y_t \Big| \to \lim_{t<s} |Y_t|$$

is an equivalence. Since the poset $S_{<s}$ has dimension at most $n-1$, this again follows from the inductive hypothesis. ☐

2.3 Application: covering pro-objects

In order to state the main result of this subsection, we fix some notation.

2.21 Recollection. Let $\mathcal{C}$ be an ∞-category and $\mathcal{P} \subset \mathrm{Fun}([1], \mathcal{C})$ a class of maps in $\mathcal{C}$. Since the walking arrow $[1]$ is a finite poset, the inclusion $\mathrm{Fun}([1], \mathcal{C}) \hookrightarrow \mathrm{Fun}([1], \mathrm{Pro}(\mathcal{C}))$ extends to an equivalence of ∞-categories

$$\mathrm{Pro}(\mathrm{Fun}([1], \mathcal{C})) \xrightarrow{\sim} \mathrm{Fun}([1], \mathrm{Pro}(\mathcal{C})) .$$

See [HTT, Proposition 5.3.5.15]. We abuse notation and write $\mathrm{Pro}(\mathcal{P}) \subset \mathrm{Fun}([1], \mathrm{Pro}(\mathcal{C}))$ for the image of $\mathrm{Pro}(\mathcal{P}) \subset \mathrm{Pro}(\mathrm{Fun}([1], \mathcal{C}))$ under this equivalence. Equivalently, a map f in $\mathrm{Pro}(\mathcal{C})$ is in $\mathrm{Pro}(\mathcal{P})$ if and only if f can be represented as a natural transformation of cofiltered diagrams $f_\bullet \colon Y_\bullet \to X_\bullet$ where each $f_s \colon Y_s \to X_s$ is in $\mathcal{P}$.

Our goal is to prove the following:

2.22 Proposition. *Let $\mathcal{C}$ be an ∞-category with finite limits, let $\mathcal{P}$ be a reasonable class of maps in $\mathcal{C}$, and let $\mathcal{B} \subset \mathcal{C}$ be a full subcategory with the property that for each $X \in \mathcal{C}$, there exists a map $Y \to X$ in $\mathcal{P}$ with $Y \in \mathcal{B}$. Then for each $X \in \mathrm{Pro}(\mathcal{C})$, there exists a map $f \colon Y \to X$ in $\mathrm{Pro}(\mathcal{P})$ with $Y \in \mathrm{Pro}(\mathcal{B})$.*

To prove Proposition 2.22, we'll use the fact that a pro-object can always be chosen to be indexed on the opposite of a filtered poset satisfying the following additional property:

2.23 Definition. A poset S is *down-finite* if for each $s \in S$, the poset $S_{\leq s}$ is finite.

2.24 Lemma [SAG, Lemma E.1.6.4]**.** *Let S' be a filtered poset. Then there exists a down-finite filtered poset S and a* colim*-equivalence $\phi \colon S \to S'$.*

Proof. Let S be the collection of all finite subsets of S' that contain a largest element, partially ordered by inclusion. Let $\phi \colon S \to S'$ be the function that sends such a subset to its largest element. By Quillen's Theorem A, we need to show that for each $s \in S$, the poset $S \times_{S'} S'_{\geq s}$ is weakly contractible. Notice that $S \times_{S'} S'_{\geq s}$ is the full subposet of S consisting of those finite subsets of S' whose largest element is $\geq s$. Note that the element $\{s\}$ is initial in $S \times_{S'} S'_{\geq s}$; hence $S \times_{S'} S'_{\geq s}$ is weakly contractible ☐

Thus Proposition 2.22 follows from the more refined statement that we can always cover a diagram in $\mathcal{C}$ indexed by the opposite of a down-finite poset by a diagram in $\mathcal{B}$. To prove this, we use the following observation.

2.25 Observation. Let S be a down-finite poset. Then the height of every element of S is finite. Hence the assignment $s \mapsto \mathrm{ht}(s)$ defines a map of posets $\mathrm{ht} \colon S \to \mathbf{N}$. In this case, the height is determined by the following requirement: for each $s \in S$, the number $\mathrm{ht}(s)$ is the smallest natural number not equal to $\mathrm{ht}(t)$ for $t < s$.

2.26 Proposition. *Let $\mathcal{C}$ be an ∞-category with finite limits, let $\mathcal{P}$ be a reasonable class of maps in $\mathcal{C}$, and let $\mathcal{B} \subset \mathcal{C}$ be a full subcategory with the property that for each $X \in \mathcal{C}$, there exists a map $Y \to X$ in $\mathcal{P}$ with $Y \in \mathcal{B}$.*

Let S be a down-finite poset. Then for every diagram $X \colon S^{\mathrm{op}} \to \mathcal{C}$, there exists a diagram $Y \colon S^{\mathrm{op}} \to \mathcal{B}$ and a natural transformation $f \colon Y \to X$ with the following properties:

(1) *For each $s \in S$, the map $f_s\colon Y_s \to X_s$ is in $\mathcal{P}$.*

(2) *For each $s \in S$, the induced map $Y_s \to X_s \times_{\lim_{t<s} X_t} \lim_{t<s} Y_s$ is in $\mathcal{P}$.*

Proof. Note that since S is down-finite,

$$S = \bigcup_{n \in \mathbf{N}} S_{\mathrm{ht} \leq n} .$$

For each $n \in \mathbf{N}$, denote the restriction of X to $S^{\mathrm{op}}_{\mathrm{ht}\leq n}$ by $X^{(n)}$. We proceed by constructing Y as an amalgam of diagrams $Y^{(n)}\colon S^{\mathrm{op}}_{\mathrm{ht}\leq n} \to \mathcal{B}$ by induction on $n \in \mathbf{N}$. For the base case $n = 0$, note that $S_{\mathrm{ht}=0}$ is the set of minimal elements of S (there are no nontrivial inequalities). For each $s \in S_{\mathrm{ht}=0}$, choose a map $f_s\colon Y_s \to X_s$ in $\mathcal{P}$ with $Y_s \in \mathcal{B}$. Then $Y^{(0)}\colon S^{\mathrm{op}}_{\mathrm{ht}=0} \to \mathcal{B}$ is given by sending s to Y_s and the natural transformation $Y^{(0)} \to X^{(0)}$ is determined by the maps $(f_s)_{s \in S^{\mathrm{op}}_{\mathrm{ht}=0}}$.

For the induction step, let $n \geq 1$ and assume that a diagram

$$Y^{(n-1)}\colon S^{\mathrm{op}}_{\mathrm{ht}\leq n-1} \to \mathcal{B}$$

and a natural transformation $f^{(n-1)}\colon Y^{(n-1)} \to X^{(n-1)}$ satisfying (1) and (2) have been constructed. Let $s \in S_{\mathrm{ht}=n}$. To define the value of Y on s, first consider the pullback

$$\begin{tikzcd} Y'_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \lim_{t<s} Y^{(n-1)}_t \arrow[d] \\ X_s \arrow[r] & \lim_{t<s} X_t , \end{tikzcd}$$

where the right-hand vertical map is induced by $f^{(n-1)}$. By Proposition 2.14 and part (2) of the inductive hypothesis, the right-hand vertical map is in $\mathcal{P}$. Since $\mathcal{P}$ is stable under pullback, the left-hand vertical map is in $\mathcal{P}$. By assumption, we can choose a map

$$e_s\colon Y_s \to Y'_s$$

in $\mathcal{P}$ with $Y_s \in \mathcal{B}$. Let f_s be the composite

$$Y_s \xrightarrow{e_s} Y'_s \longrightarrow X_s ,$$

and note that since $\mathcal{P}$ is closed under composition, $f_s \in \mathcal{P}$.

Now extend $Y^{(n-1)}$ to a functor $Y^{(n)}\colon S^{\mathrm{op}}_{\mathrm{ht}\leq n} \to \mathcal{B}$ defined on objects by sending $s \in S_{\mathrm{ht}=n}$ to Y_s. Using the specified maps

$$Y_s \xrightarrow{e_s} Y'_s \longrightarrow \lim_{t<s} Y^{(n-1)}_t ,$$

to define the functoriality of $Y^{(n)}$, we see that $Y^{(n)}$ actually defines a diagram $S^{\mathrm{op}}_{\mathrm{ht}\leq n} \to \mathcal{B}$ extending $Y^{(n-1)}$.[2] Moreover, the maps $(f_s)_{s \in S^{\mathrm{op}}_{\mathrm{ht}=n}}$ allow us to extend $f^{(n-1)}$ to a natural transformation $f^{(n)}\colon Y^{(n)} \to X^{(n)}$. By construction, the properties (1) and (2) are satisfied. □

[2] See [10, Theorem 5.12] for a detailed explanation of why this is enough to extend the diagram.

2.27 Remark. Essentially the only nontrivial point in the proof of Proposition 2.26 is an application of Proposition 2.14. Thus, in light of Remark 2.18, if $\mathcal{C}$ admits small limits and $\mathcal{P}$ is closed under small products, one can relax the assumption that S be down-finite to the assumption that for each $s \in S$, the height $\mathrm{ht}(s) = \dim(S_{\leq s})$ is finite.

Proof of Proposition 2.22. First note that X can be represented by a diagram $X : S^{\mathrm{op}} \to \mathcal{C}$, where S is a filtered poset. Moreover, by Lemma 2.24, we can without loss of generality assume that S is down-finite. The claim now follows from Proposition 2.26. □

We conclude with a few applications of Proposition 2.22 to regular ∞-categories. To do so, we begin with some remarks about effective epimorphisms of pro-objects.

2.28 Definition. Let $\mathcal{C}$ be a regular ∞-category and write eff for the class of effective epimorphisms in $\mathcal{C}$. We say that a map f in $\mathrm{Pro}(\mathcal{C})$ is a *levelwise effective epimorphism* if f is in $\mathrm{Pro}(\mathrm{eff})$.

In many examples that arise in practice, every object of $\mathcal{C}$ is truncated. In this case, the levelwise effective epimorphisms admit a better description:

2.29 Remark. Let $\mathcal{C}$ be a regular ∞-category in which every object is truncated. Then by [11, Proposition 5.2.2], $\mathrm{Pro}(\mathcal{C})$ is a regular ∞-category and the class of effective epimorphisms in $\mathrm{Pro}(\mathcal{C})$ coincides with the class of levelwise effective epimorphisms.

2.30 Corollary. *Let $\mathcal{C}$ be a regular ∞-category, and let $\mathcal{B} \subset \mathcal{C}$ be a full subcategory with the property that for each $X \in \mathcal{C}$, there exists an effective epimorphism $Y \to X$ with $Y \in \mathcal{B}$. Then for each $X \in \mathrm{Pro}(\mathcal{C})$, there exists a levelwise effective epimorphism $f : Y \to X$ with $Y \in \mathrm{Pro}(\mathcal{B})$.*

Proof. In a regular ∞-category, the class of effective epimorphisms is reasonable. So this is a special case of Proposition 2.22 □

Corollary 2.30 applies to regular ∞-categories that are determined by their full subcategories of n-truncated objects.

2.31 Recollection [11, Definition 2.3.8]**.** Let $\mathcal{C}$ be a regular ∞-category and let $n \geq -2$ be an integer. We say that $\mathcal{C}$ is *n-complicial* if for every object $X \in \mathcal{C}$, there exists an n-truncated object $Y \in \mathcal{C}_{\leq n}$ and an effective epimorphism $Y \twoheadrightarrow X$.

There are many examples of regular ∞-categories with this property. The first source is algebra:

2.32 Example. If $\mathcal{A}$ is a Grothendieck abelain category, then the connective part $\mathrm{D}(\mathcal{A})_{\geq 0}$ of the derived ∞-category of $\mathcal{A}$ is 0-complicial [SAG, Proposition C.5.3.2]. Similarly, if A is an $\mathbf{E}_1$-ring that is connective and n-truncated, then the ∞-category Mod_A of A-modules is n-complicial [SAG, Example C.5.3.5].

The second source of examples is topos theory:

2.33 Example (n-localic coherent ∞-topoi)**.** Let $n \geq 1$ be an integer and let $\mathcal{X}$ be an n-localic coherent ∞-topos. Write $\mathcal{X}^{\mathrm{coh}}_{<\infty} \subset \mathcal{X}$ for the full subcategory spanned by the truncated coherent objects (see [SAG, §A.2; 4, §§3.3 & 3.8] for the relevant definitions and background). Then $\mathcal{X}^{\mathrm{coh}}_{<\infty}$ is a bounded ∞-pretopos and $\mathcal{X}$ is naturally equivalent to the ∞-topos $\mathrm{Sh}_{\mathrm{eff}}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ of sheaves on $\mathcal{X}^{\mathrm{coh}}_{<\infty}$ for the effective epimorphism topology. Write $\mathcal{X}^{\mathrm{coh}}_{\leq n-1} \subset \mathcal{X}^{\mathrm{coh}}_{<\infty}$ for the full subcategory spanned by the $(n-1)$-truncated coherent objects. Since $\mathcal{X}$ is n-localic, restriction along the inclusion defines an equivalence of ∞-topoi

$$\mathcal{X} \simeq \mathrm{Sh}_{\mathrm{eff}}(\mathcal{X}^{\mathrm{coh}}_{<\infty}) \xrightarrow{\sim} \mathrm{Sh}_{\mathrm{eff}}(\mathcal{X}^{\mathrm{coh}}_{\leq n-1}) .$$

As a consequence, for each $X \in \mathcal{X}^{\mathrm{coh}}_{<\infty}$, there exists an effective epimorphism $Y \twoheadrightarrow X$ with $Y \in \mathcal{X}^{\mathrm{coh}}_{\leq n-1}$. That is, the regular ∞-category $\mathcal{X}^{\mathrm{coh}}_{<\infty}$ is $(n-1)$-complicial.

Now for the key consequence of Corollary 2.30:

2.34 Corollary. *Let $n \geq -2$ be an integer and let $\mathcal{C}$ be an n-complicial regular ∞-category. Then:*

(1) *For every object $X \in \mathrm{Pro}(\mathcal{C})$, there exists an n-truncated object $Y \in \mathrm{Pro}(\mathcal{C}_{\leq n})$ and a levelwise effective epimorphism $Y \twoheadrightarrow X$.*

(2) *If every object of $\mathcal{C}$ is truncated, then $\mathrm{Pro}(\mathcal{C})$ is n-complicial.*

Proof. Item (1) is a special case of Corollary 2.30, where $\mathcal{B} = \mathcal{C}_{\leq n}$ is the full subcategory spanned by the n-truncated objects. Item (2) follows from item (1) and Remark 2.29. ☐

2.35 Example. If $\mathcal{X}$ is an n-localic coherent ∞-topos, then for every object $X \in \mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$, there exists an object $Y \in \mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{\leq n-1})$ and an effective epimorphism $Y \twoheadrightarrow X$. Moreover, the regular ∞-category $\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ is $(n-1)$-complicial.

Give $\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ and $\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{\leq n-1})$ the *effective epimorphism topologies*, where covers are generated by finite jointly effectively epimorphic families of maps. Then [3, Corollary A.8] implies that restriction along the inclusion $\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{\leq n-1}) \subset \mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ defines an equivalence

$$\mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{eff}}(\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})) \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{eff}}(\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{\leq n-1}))$$

of ∞-categories of hypersheaves for effective epimorphism topology.[3]

2.36 Example. Taking $\mathcal{X} = \mathbf{An}$ and $n = 1$ in Example 2.35, we see that for every profinite anima $X \in \mathrm{Pro}(\mathbf{An}_\pi)$, there exists a profinite set Y and an effective epimorphism $Y \twoheadrightarrow X$. Moreover, restriction along the inclusion defines an equivalence

$$\mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{eff}}(\mathrm{Pro}(\mathbf{An}_\pi)) \xrightarrow{\sim} \mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{eff}}(\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}})) .$$

Here, up to set-theoretic conventions, $\mathrm{Sh}^{\mathrm{hyp}}_{\mathrm{eff}}(\mathrm{Pro}(\mathbf{Set}_{\mathrm{fin}}))$ is the definition of the ∞-category of condensed anima.

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[3]Of course, since $\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{<\infty})$ and $\mathrm{Pro}(\mathcal{X}^{\mathrm{coh}}_{\leq n-1})$ are large ∞-categories, one needs to either deal properly with unvierses or accessible sheaves in order to formulate this result completely precisely.

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