Homotopy types of schemes

Down under, August 2026

Goal of lecture series. Give a "modern" introduction to étale homotopy theory. Explain recent work refining the étale homotopy type using condensed math.

Some References. Most of what we'll cover is contained in:

> Hoyois, "Higher Galois theory"

> H-Holzschuh-Wolf, "The fundamental fiber sequence in étale homotopy theory"

> H-Holzschuh-Wolf, "Nonabelian basechange theorems & étale homotopy theory"

> Barwick-Glasman-H, "Exodromy"

> H-Holzschuh-Lara-Mair-Martini-Wolf, "The condensed homotopy type of a scheme"

Lecture 1

Notation An = ∞-category of anima / spaces / ∞-groupoids

$|-|: \mathrm{Top} \longrightarrow \mathrm{An}$ underlying anima / homotopy type

Desire of étale homotopy theory. Construct an étale homotopy type

$$\Pi_\infty^{\text{ét}}: \mathrm{Sch} \longrightarrow \mathrm{Pro}(\mathrm{An})$$

such that

$$\pi_1 \Pi_\infty^{\text{ét}}(X) \simeq \pi_1^{\text{ét}}(X)$$

there are two variants; one is a pro-group, the other is profinite

$$H^*(\Pi_\infty^{\text{ét}}(X), R) \simeq H^*_{\text{ét}}(X, R)$$

discrete ring

and $\Pi_\infty^{\text{ét}}(X)$ "satisfies all of the basic theorems in étale cohomology"

Problems

(1) $\pi_1^{\text{ét}}(X)$ is a pro-object

(2) In algebraic geometry, we don't have a reasonable space to do homotopy theory with – we have an ∞-category of sheaves.

Question that will motivate our definition. Given a reasonable topological space T, how can we recover the underlying anima $|T|$ from the ∞-category $\mathrm{Sh}(T)$ of sheaves of anima on T?

Shape theory in topology

AKA The nonabelian version of comparing singular and sheaf cohomology

Technical Digression (on hypersheaves). Let T be a space. Consider the following full subcategories of

$$\mathrm{PSh}(T) := \mathrm{Fun}(\mathrm{Open}(T)^{op}, \mathrm{An})$$

(1) The full subcategory $\mathrm{Sh}(T)$ spanned by the sheaves

(2) The full subcategory $\mathrm{Sh}^{hyp}(T)$ spanned by the presheaves that satisfy descent for hypercoverings

(3) The localization of $\mathrm{PSh}(T)$ at the stalk-wise equivalences.

Then $(2) = (3)$, and $\mathrm{Sh}^{hyp}(T) \subset \mathrm{Sh}(T)$ admits a left exact left adjoint. However, the inclusion is generally strict

Ex If T admits a CW structure, then $\mathrm{Sh}^{hyp}(T) = \mathrm{Sh}(T)$

Basis Lemma If $\mathcal{B} \subset \mathrm{Open}(T)$ is a basis for the topology on T, then the restriction functor

$$\mathrm{Sh}^{hyp}(T) \longrightarrow \mathrm{Sh}^{hyp}(\mathcal{B})$$

is an equivalence with inverse right Kan extension along the inclusion $\mathcal{B}^{op} \subset \mathrm{Open}(T)^{op}$

van Kampen Theorem The functor $|-| : \mathrm{Top} \longrightarrow \mathrm{An}$ is a hypercomplete cosheaf, i.e., for every space T and hypercovering $U_\bullet \longrightarrow T$, the natural map

$$\operatorname*{Colim}_{[n] \in \Delta^{op}} |U_n| \longrightarrow |T|$$

is an equivalence.

Construction. Let T be a space. Write $L: \mathrm{PSh}(T) \longrightarrow \mathrm{An}$ for the left Kan extension of $|-| : \mathrm{Open}(T) \longrightarrow \mathrm{An}$ along the Yoneda embedding:

$$\begin{tikzcd} \mathrm{Open}(T) \arrow[r, "|-|"] \arrow[d, hook] & \mathrm{An} \\ \mathrm{PSh}(T) \arrow[ur, "L"'] & \end{tikzcd}$$

Then L has a right adjoint R given by the formula

$$R(A)(U) = \mathrm{Map}_{\mathrm{An}}(|U|, A).$$

> By the van Kampen theorem, R factors through $\mathrm{Sh}^{\mathrm{hyp}}(T)$

Prop. If T is a locally weakly contractible space, then the functor $L: \mathrm{Sh}^{\mathrm{hyp}}(T) \longrightarrow \mathrm{An}$ is left adjoint to the constant hypersheaf functor $T^*: \mathrm{An} \longrightarrow \mathrm{Sh}^{\mathrm{hyp}}(T)$.

Proof.
Write $\mathrm{Open}_{wc}(T) \subset \mathrm{Open}(T)$ for the subposet of weakly contractible opens. Notice that the composite

basis Lem

$$\begin{array}{ccccc}
\mathrm{An} & \xrightarrow{R} & \mathrm{Sh}^{hyp}(T) & \xrightarrow[\text{rest}]{\sim} & \mathrm{Sh}^{hyp}(\mathrm{Open}_{wc}(T)) \\
A & \longmapsto & & & [U \longmapsto \mathrm{Map}(\underbrace{|U|}_{*}, A) \simeq A]
\end{array}$$

is the constant presheaf functor. But it's also a sheaf! $\square$

Remark. This implies that for a locally weakly contractible space T and abelian group M,

$$C^{-*}_{\mathrm{sing}}(T, M) \simeq R\Gamma_{\mathrm{sheaf}}(T; M).$$

And the argument is easy!

Upshot We can recover $|T|$ by applying the left adjoint to the const. hypersheaf functor to the terminal object

Idea: In algebraic geometry, we should use the same definition, replacing $\mathrm{Sh}^{hyp}(T)$ with the ∞-topos of étale sheaves on a scheme.

⚠ The constant étale sheaf functor doesn't admit a left adjoint. But it admits a **pro-left adjoint**.

Crash course on ∞-topoi

Thm. Let $\mathcal{X}$ be an ∞-category. TFAE:

(1) $\exists$ a small ∞-cat $\mathcal{C}$ and a left exact accessible localization

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, shift right, "\text{acces.}"'] & \mathrm{PSh}(\mathcal{C}) \arrow[l, shift right, "\text{lex}"'] \arrow[l, phantom, "\perp"] \end{tikzcd}$$

(2) Giraud's axioms: $\mathcal{X}$ is presentable and

(a) Coproducts are disjoint: $\forall\, U, V \in \mathcal{X}$, the square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \sqcup V \end{tikzcd}$$

is a pullback

(b) Colimits are universal: for every diagram $U_\bullet : A \to \mathcal{X}$, objects $V, W \in \mathcal{X}$ and morphisms

$$\operatorname*{colim}_{\alpha \in A} U_\alpha \longrightarrow W \longleftarrow V$$

we have

$$\operatorname*{colim}_{\alpha \in A} \left(U_\alpha \times_W V \right) \xrightarrow{\sim} \left(\operatorname*{colim}_{\alpha \in A} U_\alpha \right) \times_W V.$$

(c) Groupoid objects are effective: if $U_\bullet : \Delta^{op} \to \mathcal{X}$ is a simplicial object so that for every partition $[n] = S \cup T$ with $S \cap T = \{i\}$ a single point, then

$$\begin{tikzcd} U_n \simeq U_\bullet([n]) \arrow[r] \arrow[d] & U_\bullet(T) \arrow[d] \\ U_\bullet(S) \arrow[r] & U_\bullet(\{i\}) \simeq U_0 \end{tikzcd}$$

is a pullback,

then $U_\bullet \simeq \underbrace{\text{Čech nerve of } U_0 \longrightarrow |U_\bullet|}$

$f : X \to Y$ Čech nerve is $\cdots \; X \times_Y X \underset{pr_2}{\overset{pr_1}{\rightrightarrows}} X$

(3) $\mathcal{X}$ is presentable and colimits in $\mathcal{X}$ are **van Kampen**: the functor

$$\begin{tikzcd} \mathcal{X}^{op} \arrow[r] & \mathrm{Cat}_\infty^{large} \\ U \arrow[d] \arrow[r, mapsto] & \mathcal{X}_{/U} \\ V \arrow[r, mapsto] & \mathcal{X}_{/V} \arrow[u, "U \times_V (-)"'] \end{tikzcd}$$

preserves limits, i.e., $\mathcal{X}_{/\operatorname{colim}_{\alpha \in A} U_\alpha} \xrightarrow{\sim} \lim_{\alpha \in A^{op}} \mathcal{X}_{/U_\alpha}$.

⚠ Note that we're not saying that there is an ∞-site $(\mathcal{C},\tau)$ and an equivalence $\mathcal{X} \simeq Sh_\tau(\mathcal{C})$.
This is expected to be false!

Cor (of (3)). If $\mathcal{X}$ is an ∞-topos, then for any $U \in \mathcal{X}$, the ∞-cat $\mathcal{X}_{/U}$ is an ∞-topos.

Ex. T space, $U \subset T$ open, $h(U) \in Sh(T)$ sheaf represented by U. Then

$$Sh(T)_{/h(U)} \simeq Sh(U).$$

Ntn. For a scheme X, we write $X_{ét} = Sh(Ét_X; An)$ for the étale ∞-topos of X.

Ex. X scheme, $U \in Ét_X$, $h(U) \in X_{ét}$ sheaf represented by U.
Then

$$(X_{ét})_{/h(U)} \simeq U_{ét}.$$

Upshot For a covering $U_\bullet \to T$ or étale covering $U_\bullet \to X$ we have

$$\mathrm{Sh}(T) \xrightarrow{\sim} \lim_{[n]\in\Delta} \mathrm{Sh}(U_n) \quad \text{and} \quad X_{ét} \xrightarrow{\sim} \lim_{[n]\in\Delta} (U_n)_{ét}.$$

Def. Given ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$, a **geometric morphism** $f_*: \mathcal{X} \to \mathcal{Y}$ is a right adjoint whose left adjoint $f^*: \mathcal{Y} \to \mathcal{X}$ is left exact (= preserves finite limits).

> $\mathrm{RTop}_\infty := \infty$-topoi and geometric morphisms
> $\mathrm{LTop}_\infty := \infty$-topoi and left exact left adjoints
$\quad \cong$
$\quad (\mathrm{RTop}_\infty)^{op}$

Thm [HTT, Prop. 6.3.2.3 & Cor. 6.3.4.7] LTop_∞ has all limits and colimits. Moreover, the forgetful functor

$$\mathrm{LTop}_\infty \to \mathrm{Cat}_\infty^{\mathrm{large}}$$

preserves limits

Shape Theory

Ntn $\mathcal{X}$ ∞-topos

$\Gamma_* = \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, -) : \mathcal{X} \longrightarrow \mathrm{An}$ global sections functor

with left exact left adjoint $\Gamma^* : \mathrm{An} \longrightarrow \mathcal{X}$.

Observe The unique cofiltered limit-preserving extension of Γ^*

$$\mathrm{Pro}(\mathrm{An}) \longrightarrow \mathcal{X}$$

preserves all limits, hence has a left adjoint $\Gamma_\#$

Def

(1) The **shape** of an ∞-topos $\mathcal{X}$ is

$$\Pi_\infty(\mathcal{X}) = \Gamma_\#(1_{\mathcal{X}})$$

(2) For a scheme X, the **étale homotopy type** of X is

$$\Pi_\infty^{\text{ét}}(X) := \Pi_\infty(X_{\text{ét}}).$$

Ex. If T is a locally weakly contractible space, then

$$\Pi_\infty(\mathrm{Sh}^{\mathrm{hyp}}(T)) \simeq |T|$$

Q. What is the meaning of the shape?

Obs. For any anima A

$$\mathrm{Map}_{\mathrm{Pro}(\mathrm{An})}(\Gamma_\#(1_X), A) \simeq \mathrm{Map}_X(1_X, \Gamma^*(A))$$

$$\simeq \underbrace{\Gamma_*\Gamma^*(A)}_{\text{global sections of constant sheaf w/ coefficients in } A}$$

Obs. In $\mathrm{Pro}(\mathrm{An})$

$$\mathrm{Map}(\text{"}\lim_i\text{"} A_i, \text{"}\lim_j\text{"} B_j) \simeq \operatorname{colim}_j \mathrm{Map}(\text{"}\lim_i\text{"} A_i, B_j),$$

so a proanima is determined by mapping to all anima.

Lem. The functor

left exact

$$\mathrm{Pro}(\mathrm{An}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{lex,\,accessible}}(\mathrm{An}, \mathrm{An})^{op}$$

$$A \longmapsto \mathrm{Map}_{\mathrm{Pro}(\mathrm{An})}(A, -)$$

is an equivalence of ∞-categories

Upshot. $\Pi_\infty(X) = \Gamma_* \Gamma^* \in \mathrm{Fun}^{\mathrm{lex,acc}}(\mathrm{An}, \mathrm{An})^{op} \simeq \mathrm{Pro}(\mathrm{An})$

Functonality of the Shape

Obs. The functor

$$\begin{aligned} \mathrm{An} &\longrightarrow \mathrm{RTop}_\infty \\ A &\longmapsto \mathrm{Fun}(A, \mathrm{An}) \simeq \mathrm{An}_{/A} \end{aligned}$$ w/ right Kan extension functoriality

is left exact and fully faithful. Hence extends to a right adjoint

$$\lambda : \mathrm{Pro}(\mathrm{An}) \longrightarrow \mathrm{RTop}_\infty$$

Thm (Lurie). The left adjoint to λ is on objects given by the shape.

Corollary. For any ∞-topos X and diagram $U_\bullet : I \longrightarrow X$,

$$\operatorname*{colim}_{i \in I} \Pi_\infty(X_{/U_i}) \xrightarrow{\sim} \Pi_\infty(X_{/\operatorname*{colim}_i U_i})$$

Cor. The étale homotopy type

$$\Pi_\infty^{\text{ét}} : \mathrm{Sch} \longrightarrow \mathrm{Pro}(\mathrm{An})$$

is an étale cosheaf.