Homotopy types of schemes

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Down under, August 2026

Goal of the lecture series

Give a "modern" introduction to étale homotopy theory. Explain some recent work refining the étale homotopy type using condensed mathematics.

Some references

Most of what we will cover is contained in the following references.

- Hoyois, *Higher Galois theory* [8].
- Haine–Holzschuh–Wolf, *The fundamental fiber sequence in étale homotopy theory* [6]
- Haine–Holzschuh–Wolf, *Nonabelian basechange theorems & étale homotopy theory* [5]
- Barwick–Glasman–Haine, *Exodromy* [1].
- Haine–Holzschuh–Lara–Mair–Martini–Wolf, *The condensed homotopy type of a scheme* [4].

1 Lecture 1

1.1 Notation. We write **An** for the ∞-category of anima, spaces, or ∞-groupoids. We write

$$|-| : \mathbf{Top} \to \mathbf{An}$$

for the functor that assigns to a topological space its underlying anima or homotopy type.

The desire of étale homotopy theory is to construct an *étale homotopy type*

$$\Pi_\infty^{\text{ét}} : \mathbf{Sch} \to \mathrm{Pro}(\mathbf{An})$$

such that:

(1) $\pi_1 \Pi_\infty^{\text{ét}}(X)$ agrees with the *étale fundamental group* $\pi_1^{\text{ét}}(X)$ for X.

(2) For every (discrete) ring R, there is a natural isomorphism

$$\mathrm{H}^*(\Pi_\infty^{\text{ét}}(X); R) \simeq \mathrm{H}^*_{\text{ét}}(X; R) ,$$

between the R-cohomology of $\Pi_\infty^{\text{ét}}(X)$ and étale cohomology of X with coefficients in R.

(3) The étale homotopy type $\Pi_\infty^{\text{ét}}(X)$ “satisfies all of the basic theorems in étale cohomology.”

There are two problems.

(1) The object $\pi_1^{\text{ét}}(X)$ is a pro-object.

(2) In algebraic geometry, we do not have a reasonable space with which to do homotopy theory; we have an ∞-category of sheaves.

1.2 Question. Given a reasonable topological space T, how can we recover the underlying anima $|T|$ from the ∞-category $\mathrm{Sh}(T)$ of sheaves of anima on T?

1.1 Shape theory in topology

This is the nonabelian version of comparing singular and sheaf cohomology.

Technical digression (on hypersheaves)

Let T be a space. Consider the following full subcategories of

$$\mathrm{PSh}(T) := \mathrm{Fun}(\mathrm{Open}(T)^{\mathrm{op}}, \mathbf{An}) .$$

(1) The full subcategory $\mathrm{Sh}(T)$ is spanned by the sheaves.

(2) The full subcategory $\mathrm{Sh}^{\mathrm{hyp}}(T)$ is spanned by the presheaves that satisfy descent for hypercoverings.

(3) The third subcategory is the localization of $\mathrm{PSh}(T)$ at the stalk-wise equivalences.

The subcategories (2) and (3) are equal, and the inclusion $\mathrm{Sh}^{\mathrm{hyp}}(T) \subset \mathrm{Sh}(T)$ admits a left exact left adjoint. However, the inclusion is generally strict.

1.3 Example. If T admits a CW structure, then $\mathrm{Sh}^{\mathrm{hyp}}(T) = \mathrm{Sh}(T)$.

1.4 Lemma (basis lemma)**.** *If $\mathcal{B} \subset \mathrm{Open}(T)$ is a basis for the topology on T, then the restriction functor*

$$\mathrm{Sh}^{\mathrm{hyp}}(T) \to \mathrm{Sh}^{\mathrm{hyp}}(\mathcal{B})$$

is an equivalence with inverse given by right Kan extension along the inclusion $\mathcal{B}^{\mathrm{op}} \subset \mathrm{Open}(T)^{\mathrm{op}}$.

1.5 Theorem (van Kampen theorem)**.** *The functor $|-| : \mathbf{Top} \to \mathbf{An}$ is a hypercomplete cosheaf; that is, for every space T and hypercovering $U_\bullet \to T$, the natural map*

$$\operatorname*{colim}_{[n] \in \mathbf{\Delta}^{\mathrm{op}}} |U_n| \to |T|$$

is an equivalence.

1.6 Construction. Let T be a space. Write $L \colon \mathrm{PSh}(T) \to \mathbf{An}$ for the left Kan extension of $|-| : \mathrm{Open}(T) \to \mathbf{An}$ along the Yoneda embedding:

$$\begin{tikzcd} \mathrm{Open}(T) \arrow[r, "|-|"] \arrow[d, hook, "よ"'] & \mathbf{An} . \\ \mathrm{PSh}(T) \arrow[ur, "L"'] & \end{tikzcd}$$

Then L has a right adjoint R given by the formula

$$R(A)(U) \coloneqq \mathrm{Map}_{\mathbf{An}}(|U|, A) .$$

By Theorem 1.5, R factors through $\mathrm{Sh}^{\mathrm{hyp}}(T)$.

1.7 Proposition. *If T is a locally weakly contractible space, then the functor $L \colon \mathrm{Sh}^{\mathrm{hyp}}(T) \to \mathbf{An}$ is left adjoint to the constant hypersheaf functor $T^* : \mathbf{An} \to \mathrm{Sh}^{\mathrm{hyp}}(T)$.*

Proof. Write $\mathrm{Open}_{\mathrm{wc}}(T) \subset \mathrm{Open}(T)$ for the subposet of weakly contractible open subsets. By Lemma 1.4, the composite

$$\mathbf{An} \xrightarrow{R} \mathrm{Sh}^{\mathrm{hyp}}(T) \xrightarrow[\simeq]{\text{restrict}} \mathrm{Sh}^{\mathrm{hyp}}(\mathrm{Open}_{\mathrm{wc}}(T))$$

sends A to

$$\big[U \longmapsto \mathrm{Map}(|U|, A) \simeq A\big] .$$

This is the constant presheaf functor, but it is also a sheaf. □

1.8 Remark. This implies that, for a locally weakly contractible space T and abelian group M,

$$\mathrm{C}^*_{\mathrm{sing}}(T; M) \simeq \mathrm{R}\Gamma_{\mathrm{sheaf}}(T; M) .$$

The argument is easy.

The upshot is that we can recover $|T|$ by applying the left adjoint to the constant hypersheaf functor to the terminal object.

The idea in algebraic geometry is to use the same definition, replacing $\mathrm{Sh}^{\mathrm{hyp}}(T)$ with the ∞-topos of étale sheaves on a scheme.

1.9 Warning. The constant étale sheaf functor does not admit a left adjoint. It does, however, admit a pro-left adjoint.

1.2 Crash course on ∞-topoi

1.10 Recollection (groupoid objects)**.**

1.11 Recollection (Čech nerve)**.**

1.12 Theorem. *Let $\mathcal{X}$ be an ∞-category. The following conditions are equivalent.*

(1) *There is a small ∞-category $\mathcal{C}$ and a left exact accessible localization*

$$\mathrm{PSh}(\mathcal{C}) \underset{\longleftarrow}{\overset{\text{lex,acc}}{\longrightarrow}} \mathcal{X}$$

(2) *The ∞-category $\mathcal{X}$ is presentable, and the following conditions hold.*

  (1) Coproducts are disjoint: *for every $U, V \in \mathcal{X}$, the square*

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & U \sqcup V \end{tikzcd}$$

  is a pullback.

  (2) Colimits are universal: *for every diagram $U_\bullet \colon A \to \mathcal{X}$, objects $V, W \in \mathcal{X}$, and morphisms $\mathrm{colim}_{\alpha \in A} U_\alpha \to W \leftarrow V$, the natural map*

$$\underset{\alpha \in A}{\mathrm{colim}}(U_\alpha \times_W V) \to \left(\underset{\alpha \in A}{\mathrm{colim}}\, U_\alpha\right) \times_W V$$

  is an equivalence.

  (3) Groupoid objects are effective: *if $U_\bullet \colon \mathbf{\Delta}^{\mathrm{op}} \to \mathcal{X}$ is a simplicial object such that, for every partition $[n] = S \cup T$ with $S \cap T = \{i\}$ a single point, the square*

$$\begin{tikzcd} U_n \simeq U_\bullet([n]) \arrow[r] \arrow[d] & U_\bullet(T) \arrow[d] \\ U_\bullet(S) \arrow[r] & U_\bullet(\{i\}) \simeq U_0 \end{tikzcd}$$

  is a pullback, then $U_\bullet$ is the Čech nerve of $U_0 \to |U_\bullet|$. Recall that the Čech nerve of a morphism $f \colon X \to Y$ has the form

$$\cdots \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; X \times_Y X \; \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} \; X \,.$$

(3) *The ∞-category $\mathcal{X}$ is presentable, and colimits in $\mathcal{X}$ are van Kampen: the functor*

$$\mathcal{X}^{\mathrm{op}} \longrightarrow \mathbf{Cat}_\infty^{\mathrm{large}}$$

$$\begin{tikzcd} U \arrow[r, mapsto] \arrow[d] & \mathcal{X}_{/U} \\ V \arrow[r, mapsto] & \mathcal{X}_{/V} \arrow[u, "U \times_V (-)"'] \end{tikzcd}$$

preserves limits; that is, the natural map

$$\mathcal{X}_{/\operatorname{colim}_{i\in I} U_i} \to \lim_{i\in I^{\mathrm{op}}} \mathcal{X}_{/U_i}$$

is an equivalence.

1.13 Warning. We are not saying that there is an ∞-site $(\mathcal{C}, \tau)$ and an equivalence $\mathcal{X} \simeq \mathrm{Sh}_\tau(\mathcal{C})$. This is expected to be false.

1.14 Observation. Let $\mathcal{X}$ be an ∞-topos $U \in \mathcal{X}$. Then by (3), the ∞-category $\mathcal{X}_{/U}$ is an ∞-topos. Moreover, let $f : U \to V$ be a map in $\mathcal{X}$. Since colimits in $\mathcal{X}$ are universal, the functor

$$f^* \coloneqq U \times_V (-) : \mathcal{X}_{/V} \to \mathcal{X}_{/U}$$

preserves colimits. Hence admits a right adjoint $f_* : \mathcal{X}_{/U} \to \mathcal{X}_{/V}$. Also note that f^* is right adjoint to the forgetful functor, which we denote by

$$f_\sharp : \mathcal{X}_{/U} \to \mathcal{X}_{/V} .$$

In particular, f_* is a geometric morphism.

1.15 Example. Let T be a space, let $U \subset T$ be open, and let $よ(U) \in \mathrm{Sh}(T)$ be the sheaf represented by U. Then

$$\mathrm{Sh}(T)_{/よ(U)} \simeq \mathrm{Sh}(U) .$$

1.16 Notation. For a scheme X, write

$$X_{\text{ét}} \coloneqq \mathrm{Sh}(\mathrm{Et}_X; \mathbf{An})$$

for the étale ∞-topos of X.

1.17 Example. Let X be a scheme, let $U \in \mathrm{Et}_X$, and let $よ(U) \in X_{\text{ét}}$ be the sheaf represented by U. Then

$$(X_{\text{ét}})_{/よ(U)} \simeq U_{\text{ét}} .$$

The upshot is that, for a covering $U_\bullet \to T$, or an étale covering $U_\bullet \to X$, we have equivalences

$$\mathrm{Sh}(T) \xrightarrow{\simeq} \lim_{[n]\in\mathbf{\Delta}} \mathrm{Sh}(U_n) ,$$

and

$$X_{\text{ét}} \xrightarrow{\simeq} \lim_{[n]\in\mathbf{\Delta}} (U_n)_{\text{ét}} .$$

1.18 Definition. Given ∞-topoi $\mathcal{X}$ and $\mathcal{Y}$, a *geometric morphism* $f_* : \mathcal{X} \to \mathcal{Y}$ is a right adjoint whose left adjoint $f^* : \mathcal{Y} \to \mathcal{X}$ is left exact, meaning that it preserves finite limits.

Write

$$\mathbf{RTop}_\infty \coloneqq \{\infty\text{-topoi and geometric morphisms}\} ,$$

and

$$\mathbf{LTop}_\infty \coloneqq \{\infty\text{-topoi and left exact left adjoints}\} .$$

There is an equivalence

$$(\mathbf{RTop}_\infty)^{\mathrm{op}} \simeq \mathbf{LTop}_\infty .$$

1.19 Theorem (HTT, Proposition 6.3.2.3 and Corollary 6.3.4.7)**.** *The ∞-category $\mathbf{LTop}_\infty$ has all limits and colimits. Moreover, the forgetful functor*

$$\mathbf{LTop}_\infty \to \mathbf{Cat}_\infty^{\mathrm{large}}$$

preserves limits.

2 Lecture 2

2.1 Shape theory

2.1 Notation. Let $\mathcal{X}$ be an ∞-topos. The *global sections functor* is the functor

$$\Gamma_* := \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, -) \colon \mathcal{X} \to \mathbf{An}\,.$$

It admits a left exact left adjoint $\Gamma^* \colon \mathbf{An} \to \mathcal{X}$.

2.2 Recollection. Let $\mathcal{C}$ be an ∞-category. The ∞-category $\mathrm{Pro}(\mathcal{C})$ of pro-objects in $\mathcal{C}$ is an ∞-category equipped with a functor $よ \colon \mathcal{C} \to \mathrm{Pro}(\mathcal{C})$ satisfying the following universal property.

(1) The ∞-category $\mathrm{Pro}(\mathcal{C})$ admits cofiltered limits.

(2) If $\mathcal{D}$ is an ∞-category with cofiltered limits, then restriction induces an equivalence

$$\mathrm{Fun}^{\mathrm{cofil}}(\mathrm{Pro}(\mathcal{C}), \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{D})\,.$$

If $\mathcal{C}$ has finite limits, then $\mathrm{Pro}(\mathcal{C})$ has all limits, and $よ$ preserves finite limits.

2.3 Observation. For an ∞-topos $\mathcal{X}$, since Γ^* is left exact, the unique cofiltered-limit-preserving extension

$$\mathrm{Pro}(\mathbf{An}) \to \mathcal{X}$$

of Γ^* preserves all limits. Hence, by an appropriate version of the adjoint functor theorem, this functor admits a left adjoint

$$\Gamma_\sharp \colon \mathcal{X} \to \mathrm{Pro}(\mathbf{An})\,.$$

2.4 Definition.

(1) The *shape* of an ∞-topos $\mathcal{X}$ is

$$\Pi_\infty(\mathcal{X}) := \Gamma_\sharp(1_{\mathcal{X}})\,.$$

(2) For a scheme X, the *étale homotopy type* of X is

$$\Pi_\infty^{\text{ét}}(X) := \Pi_\infty(X_{\text{ét}})\,.$$

2.5 Example. If T is a locally weakly contractible space, then

$$\Pi_\infty(\mathrm{Sh}^{\mathrm{hyp}}(T)) \simeq |T|\,\,.$$

2.6 Example. If T is the Warsaw circle, then

$$\Pi_\infty(\mathrm{Sh}(T)) \simeq \mathrm{S}^1$$

as a constant pro-anima.

2.7 Remark. There is a classical theory of shapes of compact Hausdorff spaces. The shape of T in the classical sense coincides with $\Pi_\infty(\mathrm{Sh}(T))$.

2.8 Example. If $\mathcal{C}$ is a small ∞-category, then the constant functor $\Gamma^* \colon \mathbf{An} \to \mathrm{Fun}(\mathcal{C}, \mathbf{An})$ admits a genuine left adjoint $\mathrm{colim}_{\mathcal{C}}$. Hence

$$\Pi_\infty(\mathrm{Fun}(\mathcal{C}, \mathbf{An})) \simeq \mathrm{colim}_{\mathcal{C}} * \simeq \mathrm{B}\mathcal{C} ,$$

where $\mathrm{B}\mathcal{C}$ is the classifying anima of $\mathcal{C}$. Equivalently, the functor $\mathrm{B} \colon \mathbf{Cat}_\infty \to \mathbf{An}$ is left adjoint to the inclusion $\mathbf{An} \subset \mathbf{Cat}_\infty$.

2.9 Question. What is the meaning of the shape?

2.10 Observation. For any anima A,

$$\begin{aligned} \mathrm{Map}_{\mathrm{Pro}(\mathbf{An})}(\Gamma_\sharp(1_{\mathcal{X}}), A) &\simeq \mathrm{Map}_{\mathcal{X}}(1_{\mathcal{X}}, \Gamma^*(A)) \\ &\simeq \Gamma_*\Gamma^*(A) . \end{aligned}$$

The final expression is the anima of global sections of the constant sheaf with coefficients in A.

2.11 Observation. In $\mathrm{Pro}(\mathbf{An})$,

$$\mathrm{Map}\left(\text{“}\lim_i\text{”}\, A_i, \text{“}\lim_j\text{”}\, B_j\right) \simeq \mathrm{colim}_j\, \mathrm{Map}\left(\text{“}\lim_i\text{”}\, A_i, B_j\right) .$$

Thus a pro-anima is determined by mapping to all anima.

2.12 Lemma. *The functor*

$$\begin{aligned} \mathrm{Pro}(\mathbf{An}) &\longrightarrow \mathrm{Fun}^{\mathrm{lex,acc}}(\mathbf{An}, \mathbf{An})^{\mathrm{op}} \\ A &\longmapsto \mathrm{Map}_{\mathrm{Pro}(\mathbf{An})}(A, -) \end{aligned}$$

is an equivalence of ∞-categories.

The upshot is that

$$\Pi_\infty(\mathcal{X}) = \Gamma_*\Gamma^* \in \mathrm{Fun}^{\mathrm{lex,acc}}(\mathbf{An}, \mathbf{An})^{\mathrm{op}} \simeq \mathrm{Pro}(\mathbf{An}) .$$

2.2 Functoriality of the shape

2.13 Observation. The functor

$$\begin{aligned} \mathbf{An} &\longrightarrow \mathbf{RTop}_\infty \\ A &\longmapsto \mathrm{Fun}(A, \mathbf{An}) \simeq \mathbf{An}_{/A} \end{aligned}$$

with right Kan extension functoriality is left exact and fully faithful. Hence it extends to a right adjoint

$$\lambda \colon \mathrm{Pro}(\mathbf{An}) \to \mathbf{RTop}_\infty .$$

2.14 Theorem (Lurie)**.** *The left adjoint to λ is, on objects, given by the shape.*

2.15 Corollary. *For any ∞-topos $\mathcal{X}$ and diagram $U_\bullet \colon I \to \mathcal{X}$, the natural map*

$$\operatorname*{colim}_{i\in I} \Pi_\infty(\mathcal{X}_{/U_i}) \to \Pi_\infty\left(\mathcal{X}_{/\operatorname{colim}_{i\in I} U_i}\right)$$

is an equivalence.

2.16 Corollary. *The étale homotopy type*

$$\Pi_\infty^{\text{ét}} \colon \mathbf{Sch} \to \operatorname{Pro}(\mathbf{An})$$

is an étale cosheaf.

2.3 Locally constant objects and monodromy

2.17 Definition. Let $\mathcal{X}$ be an ∞-topos. An object $L \in \mathcal{X}$ is *locally constant* if there exist an effective epimorphism

$$\coprod_{i\in I} U_i \to 1_{\mathcal{X}}$$

and anima $(A_i)_{i\in I}$ such that there are equivalences

$$L \times U_i \simeq \Gamma^*_{\mathcal{X}_{/U_i}}(A_i) \simeq \Gamma^*_{\mathcal{X}}(A_i) \times U_i$$

in $\mathcal{X}_{/U_i}$. We say that L is *lisse* if, in addition, I can be chosen to be finite and each A_i can be chosen to be π-finite. We write

$$\mathcal{X}^{\text{lisse}} \subset \mathcal{X}^{\text{lc}} \subset \mathcal{X}$$

for the full subcategories of lisse and locally constant objects.

2.18 Theorem (monodromy v1; Lurie). *Let $\mathcal{X}$ be an ∞-topos. If $\Gamma^* \colon \mathbf{An} \to \mathcal{X}$ admits a genuine left adjoint $\Gamma_\sharp \colon \mathcal{X} \to \mathbf{An}$, then the right adjoint to*

$$\Gamma_\sharp \colon \mathcal{X} \longrightarrow \mathbf{An}_{/\Gamma_\sharp(1_{\mathcal{X}})} \simeq \operatorname{Fun}(\Pi_\infty(\mathcal{X}), \mathbf{An})$$

is fully faithful with image $\mathcal{X}^{\text{lc}}$. That is, there is an equivalence

$$\mathcal{X}^{\text{lc}} \simeq \operatorname{Fun}(\Pi_\infty(\mathcal{X}), \mathbf{An})\ .$$

2.19 Example. If T is a locally weakly contractible space, then

$$\operatorname{LC}^{\text{hyp}}(T) \simeq \operatorname{Fun}(|T|, \mathbf{An})\ .$$

For lisse objects, we do not need any assumptions on $\mathcal{X}$. We first need some technical digressions.

Background on proanima

2.20 Observation. The adjunctions

$$\mathbf{An} \underset{\longleftarrow}{\overset{\tau_{\leq n}}{\longrightarrow}} \mathbf{An}_{\leq n}$$

induce adjunctions

$$\operatorname{Pro}(\mathbf{An}) \underset{\longleftarrow}{\overset{\tau_{\leq n}}{\longrightarrow}} \operatorname{Pro}(\mathbf{An}_{\leq n})\ .$$

2.21 Warning. It is possible that a map of proanima $f : X \to Y$ induces an equivalence on every $\tau_{\leq n}$ but is not an equivalence. If $A \in \mathbf{An}$ is regarded as a constant pro-anima and

$$\tau_{<\infty} A \coloneqq \left\{\cdots \to \tau_{\leq n+1} A \to \tau_{\leq n} A \to \cdots\right\} ,$$

then there is a natural map $A \to \tau_{<\infty} A$ that is an equivalence on every $\tau_{\leq n}$. However,

$$\mathrm{Map}_{\mathrm{Pro}(\mathbf{An})}(\tau_{<\infty} A, A) \simeq \operatorname*{colim}_{n\to\infty} \mathrm{Map}(\tau_{\leq n} A, A) .$$

Thus every map $\tau_{<\infty} A \to A$ factors through some $\tau_{\leq n} A$. Consequently, if A is not n-truncated for any n, then $A \to \tau_{<\infty} A$ cannot have an inverse.

Fact. The inclusion $\mathrm{Pro}(\mathbf{An}_{<\infty}) \subset \mathrm{Pro}(\mathbf{An})$ admits a left adjoint $\tau_{<\infty}$ that identifies

$$\mathrm{Pro}(\mathbf{An})\left[\left\{\text{equivalences on all } \tau_{\leq n}\right\}^{-1}\right] \simeq \mathrm{Pro}(\mathbf{An}_{<\infty}) .$$

We call $\tau_{<\infty}$ the *protruncation functor*.

2.22 Definition. The ∞-category of *profinite anima* is $\mathrm{Pro}(\mathbf{An}_\pi)$. The inclusion $\mathrm{Pro}(\mathbf{An}_\pi) \subset \mathrm{Pro}(\mathbf{An})$ admits a left adjoint

$$X \mapsto X^\wedge_\pi$$

called *profinite completion*.

The following facts hold.

(1) If S is a set, then

$$S^\wedge_\pi \simeq \beta(S) .$$

The right-hand side is the Čech–Stone compactification of S. Since β does not preserve finite products, profinite completion generally does not preserve them either.

(2) If A is a connected anima, then $A^\wedge_\pi$ connected and

$$\pi_1(A^\wedge_\pi) \simeq \widehat{\pi_1(A)} .$$

The right-hand side is the profinite completion of groups.

(3) In general, there exist discrete groups G for which $(\mathrm{B}G)^\wedge_\pi$ has higher homotopy. The natural map

$$(\mathrm{B}G)^\wedge_\pi \to \mathrm{B}\widehat{G}$$

is an equivalence if and only if G is *good* in the sense of Serre: for every finite G-module M, the natural map

$$\mathrm{H}^*(\widehat{G}; M) \to \mathrm{H}^*(G; M)$$

is an isomorphism.

Monodromy for lisse objects

2.23 Notation. Given an ∞-category $\mathcal{D}$, write

$$\mathrm{Fun}^{\mathrm{cts}}(-, \mathcal{D}) \colon \ \mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}} \to \mathbf{Cat}_\infty$$

for the unique filtered colimit-preserving extension of $\mathrm{Fun}(-, \mathcal{D}) \colon \ \mathbf{Cat}_\infty^{\mathrm{op}} \to \mathbf{Cat}_\infty$. Thus

$$\mathrm{Fun}^{\mathrm{cts}}\left(\text{“}\lim_{i \in I}\text{”}\, \mathcal{C}_i, \mathcal{D} \right) \simeq \operatorname*{colim}_{i \in I^{\mathrm{op}}} \mathrm{Fun}(\mathcal{C}_i, \mathcal{D}) \, .$$

2.24 Theorem (monodromy)**.** *Let $\mathcal{X}$ be an ∞-topos. There are natural equivalences*

$$\begin{aligned} \mathcal{X}^{\mathrm{lisse}} &\simeq \mathrm{Fun}^{\mathrm{cts}}(\Pi_\infty(\mathcal{X}), \mathbf{An}_\pi) \\ &\simeq \mathrm{Fun}^{\mathrm{cts}}(\Pi_\infty(\mathcal{X})^\wedge_\pi, \mathbf{An}_\pi) \, . \end{aligned}$$

Slogan. The profinite shape classifies lisse local systems.
In fact, we have the following.

2.25 Theorem (Lurie)**.** *The functor*

$$\lambda \colon \ \mathrm{Pro}(\mathbf{An}_\pi) \to \mathbf{RTop}_\infty$$

is fully faithful with explicit image.

2.4 Basic properties of the étale homotopy type

2.26 Convention. From now on we make the convention that for a scheme X, we simply write $\Pi_\infty^{\text{ét}}(X)$ for the protruncation of the étale homotopy type.

2.27 Example. If k is a field and $\bar{k} \supset k$ is a separable closure, then there is an equivalence

$$\widehat{\Pi}_\infty^{\text{ét}}(\mathrm{Spec}(k)) \simeq \mathrm{BGal}(\bar{k}/k) \, .$$

2.28 Example. There is an equivalence

$$\widehat{\Pi}_\infty^{\text{ét}}(\mathrm{Spec}(\mathbf{Z})) \simeq * \, .$$

Indeed, $\mathrm{Spec}(\mathbf{Z})$ has no unramified étale covers, so $\pi_1^{\text{ét}}(\mathrm{Spec}(\mathbf{Z})) = 0$. Arithmetic duality implies that $\mathrm{H}^{\geq 1}_{\text{ét}}(\mathrm{Spec}(\mathbf{Z}); \mathbf{Z}/n) = 0$. The Hurewicz theorem completes the proof.

2.29 Theorem (profiniteness)**.** *If X is a noetherian geometrically unibranch scheme (e.g., X is smooth over a field), then $\Pi_\infty^{\text{ét}}(X)$ is profinite.*

2.30 Theorem (fundamental fiber sequence)**.** *Let X be a qcqs scheme over a field k, and let $\bar{k} \supset k$ be a separable closure. There are natural fiber sequences*

$$\Pi_\infty^{\text{ét}}(X_{\bar{k}}) \to \Pi_\infty^{\text{ét}}(X) \to \mathrm{BGal}(\bar{k}/k)$$

and

$$\widehat{\Pi}_\infty^{\text{ét}}(X_{\bar{k}}) \to \widehat{\Pi}_\infty^{\text{ét}}(X) \to \mathrm{BGal}(\bar{k}/k) \, .$$

2.31 Recollection (analytification)**.** Let X be a scheme locally of finite type over $\mathbf{C}$. Applying the shape to the geometric morphism $a_* \colon \mathrm{Sh}(X(\mathbf{C})) \to X_{\text{ét}}$ together with Example 2.5, we obtain a natural map

$$|X(\mathbf{C})| \to \Pi_\infty^{\text{ét}}(X)\,.$$

2.32 Theorem (Riemann existence)**.** *Let X be a scheme of finite type over $\mathbf{C}$. The natural*

$$|X(\mathbf{C})| \to \Pi_\infty^{\text{ét}}(X)$$

induces an equivalence after profinite completions.

2.33 Theorem (Künneth formulas)**.** *Let k be a field with absolute Galois group G_k, and let X and Y be qcqs schemes over k.*

(1) *If Y is proper over k, then*

$$\widehat{\Pi}_\infty^{\text{ét}}(X \times_k Y) \simeq \widehat{\Pi}_\infty^{\text{ét}}(X) \underset{\mathrm{BG}_k}{\times} \widehat{\Pi}_\infty^{\text{ét}}(Y)\,.$$

(2) *Let $p = \mathrm{char}(k)$. If G_k is prime to p, then*

$$\widehat{\Pi}_\infty^{\text{ét}}(X \times_k Y)^{\wedge}_{p'} \simeq \widehat{\Pi}_\infty^{\text{ét}}(X)^{\wedge}_{p'} \underset{\mathrm{BG}_k}{\times} \widehat{\Pi}_\infty^{\text{ét}}(Y)^{\wedge}_{p'}\,.$$

Here $(-)^{\wedge}_{p'}$ denotes completion away from p.

2.5 Application: anabelian geometry

2.34 Theorem (A. Schmidt–Stix)**.** *Let $k \supset \mathbf{Q}$ be a finitely generated field extension, and let X and Y be smooth k-varieties. Suppose that there exist locally closed embeddings of X and Y into products of hyperbolic curves. The natural map*

$$\mathrm{Isom}_k(X, Y) \to \pi_0\,\mathrm{Equiv}_{\mathrm{BG}_k}(\widehat{\Pi}_\infty^{\text{ét}}(X), \widehat{\Pi}_\infty^{\text{ét}}(Y))$$

is a split injection with a natural retraction.

2.35 Corollary. *If $\widehat{\Pi}_\infty^{\text{ét}}(X) \simeq \widehat{\Pi}_\infty^{\text{ét}}(Y)$ over BG_k, then $X \simeq Y$ as k-schemes.*

2.36 Theorem (Holschbach–J. Schmidt–Stix)**.** *Let k be an algebraically closed field of positive characteristic, and let X be a smooth k-variety. Then $\widehat{\Pi}_\infty^{\text{ét}}(X) \simeq *$ if and only if $X \simeq \mathrm{Spec}(k)$.*

2.37 Theorem (Achinger, 2017)**.** *If X is an affine connected $\mathbf{F}_p$-scheme, then its profinite étale homotopy type is 1-truncated.*

See Holzschuh's talk for more applications.

2.6 Application: the Artin–Tate pairing

Setup. Let X be a smooth geometrically connected surface over $\mathbf{F}_q$ of dimension $2d$, and let $\ell \nmid q$ be a prime. Write

$$\mathrm{Br}_{\mathrm{nd}}(X) \coloneqq \frac{\mathrm{Br}(X)}{\{\text{divisible elements}\}}\,.$$

Milne, Artin, and Tate defined a pairing

$$\mathrm{Br}_{\mathrm{nd}}(X)[\ell^\infty] \times \mathrm{Br}_{\mathrm{nd}}(X)[\ell^\infty] \to \mathbf{Q}/\mathbf{Z}\,.$$

It is really defined on

$$\mathrm{H}^{2d}_{\text{ét}}(X; \mathbf{Q}_\ell/\mathbf{Z}_\ell(d)) \simeq \mathrm{H}^{2d+1}_{\text{ét}}(X; \mathbf{Z}_\ell(d))_{\mathrm{tors}}\,.$$

2.38 Conjecture (Tate, 1966)**.** *The Artin–Tate pairing is alternating.*

History. In 1977 and 1986, Manin gave counterexamples. In 1996, Urabe found mistakes in Manin's work.

2.39 Theorem (Feng, 2017)**.** *Tate's conjecture is true.*

The proof uses analogies from arithmetic topology. Because X is a surface, we should think of the basechange to the separable closure $X_{\overline{\mathbf{F}}_q}$ as a real 4-manifold. Because of the fiber sequence

$$\widehat{\Pi}^{\text{ét}}_\infty(X_{\overline{\mathbf{F}}_q}) \longrightarrow \widehat{\Pi}^{\text{ét}}_\infty(X) \longrightarrow \mathrm{B}\widehat{\mathbf{Z}} \simeq (\mathrm{S}^1)^\wedge_\pi\,,$$

the profinite étale homotopy type $\widehat{\Pi}^{\text{ét}}_\infty(X)$ behaves like a real $(4d+1)$-manifold. The Artin–Tate pairing corresponds to the *linking form* on an orientable $(4d+1)$-manifold. Feng used the étale homotopy type to construct Steenrod operations on étale cohomology and analogies from topology to prove the conjecture.

3 Lecture 3

Goal. We will discuss two refinements of the étale homotopy type.

(1) The first is to an object, namely a pro-1-category, which classifies constructible étale sheaves. Is from our work with Barwick and Glasman [1]; the first version came out in 2018.

(2) The second is to a condensed anima whose fundamental group recovers Bhatt and Scholze's proétale fundamental group. This is from our work with Holzschuh–Lara–Mair–Martini–Wolf from 2025.

It turns out that these generalizations are intimately related.

Motivation for the second refinement. Let X be a topologically noetherian scheme, and let $\bar{x} \to X$ be a geometric point. In their work on the proétale topology, Bhatt and Scholze introduced a topological group

$$\pi_1^{\text{proét}}(X, \bar{x})$$

called the proétale fundamental group, with the following key properties.

(1) The profinite completion of $\pi_1^{\text{proét}}(X)$ is the usual profinite étale fundamental group. However, if $\bar{k}$ is a separably closed field, then

$$\pi_1^{\text{proét}}(\mathbf{P}^1_{\bar{k}}/0 \sim \infty) \simeq \mathbf{Z}\,.$$

(2) If X is normal, then

$$\pi_1^{\text{proét}}(X) \simeq \pi_1^{\text{ét}}(X)\,.$$

(3) The group $\pi_1^{\text{proét}}(X)$ classifies $\mathbf{Q}_\ell$-local systems.

(4) In general, $\pi_1^{\text{proét}}(X)$ is not the limit of a pro-group. Additionally, there is a natural geometric morphism

$$X_{\text{proét}} \to X_{\text{ét}}$$

that induces an equivalence on protruncated shapes. Thus $\pi_1^{\text{proét}}$ cannot be accessed via shape theory in the form we have been discussing.

3.1 A quick overview of exodromy

This is the simplest case of exit-path categories in topology.

3.1 Notation. The functor

$$\mathrm{B}\colon \mathbf{Cat}_\infty \to \mathbf{An}$$

is left adjoint to the inclusion.

There are two ideas. Let T be a topological space admitting a triangulation with poset of simplices P.

(1) There is an equivalence

$$\{\text{sheaves on } T \text{ constructible with respect to the triangulation}\} \simeq \mathrm{Fun}(P, \mathbf{An})\,.$$

(2) There is an equivalence

$$|T| \simeq \mathrm{B}P\,.$$

General theory. The general theory is that of exit-path ∞-categories. See Lurie [HA, Appendix A], Clausen–Jansen [2], and H.–Porta–Teyssier [7].

3.2 Example. Let $(\mathrm{B}\mathbf{Z})^{\triangleleft}$ be the category

$$\bullet \longrightarrow \bullet \circlearrowright \mathbf{Z}$$

obtained by $\mathrm{B}\mathbf{Z}$ by adjoining an initial object. Constructible sheaves on a cone on a circle stratified by the cone point and it complement are $\mathrm{Fun}((\mathrm{B}\mathbf{Z})^{\triangleleft}, \mathbf{An})$.

Idea. We should have an algebraic geometry version of this, but we will deal with all stratifications at once.

3.3 Notation. Write $\mathbf{Cat}_{\infty,\pi}$ for the ∞-categories with finitely many objects up to equivalence and π-finite mapping anima. Then

$$\mathbf{Cat}_{\infty,\pi} \cap \mathbf{Cat}_1 = \{\text{finite 1-categories}\}\,.$$

3.4 Definition. Let $\mathcal{X}$ be an ∞-topos. The ∞-category of points of $\mathcal{X}$ is

$$\mathrm{Pt}(\mathcal{X}) \coloneqq \mathrm{Fun}^*(\mathcal{X}, \mathbf{An})\,.$$

Here the superscript $*$ denotes the left exact left adjoints.

3.5 Recollection. Let X be a qcqs scheme. The Grothendieck school proved that the category $\mathrm{Pt}(X_{\text{ét}})$ of points of the étale ∞-topos of X has the following description.

(1) An object of $\mathrm{Pt}(X_{\text{ét}})$ is a *geometric point* $\bar{x} \to X$. That is, a choice of a point $x \in X$ together with a choice of separable closure $\kappa(\bar{x}) \supset \kappa(x)$ of the residue field $\kappa(x)$ at x.

(2) Given geometric points $\bar{s} \to X$ and $\bar{\eta} \to X$, its morphisms are étale specializations, and

$$\mathrm{Hom}_{\mathrm{Pt}(X_{\text{ét}})}(\bar{s}, \bar{\eta}) = \mathrm{Hom}_X\left(\mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\bar{\eta}}), \mathrm{Spec}(\mathcal{O}^{\mathrm{sh}}_{X,\bar{s}})\right) .$$

In particular,

$$\mathrm{Hom}_{\mathrm{Pt}(X_{\text{ét}})}(\bar{x}, \bar{x}) \simeq \mathrm{Gal}(\kappa(\bar{x})/\kappa(x)) .$$

3.6 Theorem (étale homotopy type via exodromy; Barwick–Glasman–H.)**.** *Let X be a qcqs scheme. There is a natural profinite* 1*-category* $\mathrm{Gal}(X)$ *such that the following properties hold.*

(1) *The limit functor*

$$\lim : \ \mathrm{Pro}(\mathbf{Cat}_\infty) \to \mathbf{Cat}_\infty ,$$

carries $\mathrm{Gal}(X)$ *to* $\mathrm{Pt}(X_{\text{ét}})$*.*

(2) *The composite*

$$\mathrm{Pro}(\mathbf{Cat}_\infty) \xrightarrow{\mathrm{Pro(B)}} \mathrm{Pro}(\mathbf{An}) \xrightarrow{\tau_{<\infty}} \mathrm{Pro}(\mathbf{An}_{<\infty}) .$$

carries $\mathrm{Gal}(X)$ *to* $\Pi^{\text{ét}}_\infty(X)$*.*

(3) *There is a natural equivalence*

$$X^{\mathrm{cons}}_{\text{ét}} \simeq \mathrm{Fun}^{\mathrm{cts}}(\mathrm{Gal}(X), \mathbf{An}_\pi) .$$

Here the left-hand side is the category of constructible étale sheaves of anima on X: étale sheaves F for which there exists a finite stratification of X by qcqs subschemes $(Y_p)_{p\in P}$ such that each restriction $F|_{Y_i}$ is lisse.

3.2 Interlude on condensed mathematics

Setup. The category **Comp** of compact Hausdorff spaces admits a Grothendieck topology in which the coverings are the finite jointly surjective families. Every $K \in \mathbf{Comp}$ admits a surjection from a projective object. Gleason showed that the projectives are the compacta K such that, for every open $U \subset K$, the closure $\overline{U}$ is open. These are the *extremally disconnected spaces*; write **Extr** for their category. On **Extr**, every cover splits, so the basis lemma gives equivalences

$$\mathrm{Sh}^{\mathrm{hyp}}(\mathbf{Comp}) \simeq \mathrm{Sh}^{\mathrm{hyp}}(\mathbf{Extr}) \simeq \mathrm{Fun}^{\times}(\mathbf{Extr}^{\mathrm{op}}, \mathbf{An}) .$$

The right-most ∞-category consists of the functors F such that $F(\varnothing) \simeq *$ and for all extremally disconnected compacta K and L, the natural map

$$F(K \sqcup L) \to F(K) \times F(L)$$

is an equivalence.

3.7 Definition. Let $\mathcal{C}$ be an ∞-category with finite products. The ∞-category of *condensed objects* in $\mathcal{C}$ is

$$\mathrm{Cond}(\mathcal{C}) \coloneqq \mathrm{Fun}^{\times}(\mathbf{Extr}^{\mathrm{op}}, \mathcal{C}) .$$

3.8 Proposition. *Consider the functor*

$$\mathrm{Pro}(\mathbf{Cat}_{\infty}) \to \mathrm{Cond}(\mathbf{Cat}_{\infty})$$

given by extending the constant sheaf functor $\mathbf{Cat}_{\infty} \to \mathrm{Cond}(\mathbf{Cat}_{\infty})$ *along cofiltered limits. Then the restriction*

$$\mathrm{Pro}(\mathbf{Cat}_{\infty,\pi}) \subset \mathrm{Pro}(\mathbf{Cat}_{\infty}) \to \mathrm{Cond}(\mathbf{Cat}_{\infty})$$

is fully faithful.

3.9 Notation. Write

$$\mathrm{B}^{\mathrm{cond}} \colon \mathrm{Cond}(\mathbf{Cat}_{\infty}) \to \mathrm{Cond}(\mathbf{An})$$

for the left adjoint to the inclusion, given by

$$\mathcal{C} \mapsto [K \mapsto \mathrm{B}(\mathcal{C}(K))] \ .$$

3.10 Definition. Let X be a qcqs scheme. The *condensed homotopy type* of X is

$$\Pi_{\infty}^{\mathrm{cond}}(X) \coloneqq \mathrm{B}^{\mathrm{cond}}\,\mathrm{Gal}(X) .$$

In particular,

$$\Pi_{\infty}^{\mathrm{cond}}(X)(*) = \mathrm{B}(\mathrm{Pt}(X_{\text{ét}})) .$$

3.11 Theorem (Wolf)**.** *Write* $\mathbf{Cond}(\mathbf{An})$ *for the condensed* ∞*-category*

$$K \mapsto \mathrm{Cond}(\mathbf{An})_{/K} .$$

For a qcqs scheme X*, there is a natural equivalence*

$$X_{\text{proét}}^{\mathrm{hyp}} \simeq \mathrm{Fun}^{\mathrm{cond}}(\mathrm{Gal}(X), \mathbf{Cond}(\mathbf{An})) .$$

3.12 Remark. Given a geometric morphism of ∞-topoi $f_* \colon \mathcal{X} \to \mathcal{Y}$, the functor $f^* \colon \mathcal{Y} \to \mathcal{X}$ always admits a pro-left adjoint

$$f_{\sharp} \colon \mathcal{X} \to \mathrm{Pro}(\mathcal{Y}) .$$

Thus one can always talk about the *relative shape*

$$\Pi_{\infty}(\mathcal{X}/\mathcal{Y}) \coloneqq f_{\sharp}(1_{\mathcal{X}}) .$$

It follows from Wolf's theorem that $\Pi_{\infty}^{\mathrm{cond}}(X)$ coincides with the relative shape of $X_{\text{proét}}^{\mathrm{hyp}}$ over $\mathrm{Cond}(\mathbf{An})$.

3.3 Elementary properties and computations

3.13 Example. For a field k, there is an equivalence

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(k)) \simeq \mathrm{BG}_k \ .$$

More generally, if A is a henselian local ring with residue field k, then

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(A)) \simeq \mathrm{BG}_k \ .$$

3.14 Proposition (fundamental fiber sequence)**.** *Let k be a field with separable closure $\bar{k}$, and let X be a qcqs scheme over k. Then the naturally null sequence*

$$\Pi_\infty^{\mathrm{cond}}(X_{\bar{k}}) \to \Pi_\infty^{\mathrm{cond}}(X) \to \mathrm{BG}_k$$

is a fiber sequence.

3.15 Theorem. *For a compact Hausdorff space K, there is a natural equivalence*

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(\mathrm{C}(K, \mathbf{C}))) \simeq K \ ,$$

where the right-hand side is the condensed set represented by K.

3.16 Corollary. *For a compact Hausdorff space K, there is a natural equivalence*

$$\Pi_\infty^{\text{ét}}(\mathrm{Spec}(\mathrm{C}(K, \mathbf{C}))) \simeq \tau_{<\infty} \Pi_\infty(\mathrm{Sh}(K)) \ .$$

That is, up to protruncation, $\Pi_\infty^{\text{ét}}(\mathrm{Spec}(\mathrm{C}(K, \mathbf{C})))$ recovers the shape of the ∞-topos $\mathrm{Sh}(K)$*.*

3.17. By Example 2.5, if K is locally weakly contractible, then $\tau_{<\infty} \Pi_\infty(\mathrm{Sh}(K)) \simeq |K|$. In particular, for such K,

$$\pi_1^{\mathrm{SGA3}}(\mathrm{Spec}(\mathrm{C}(K, \mathbf{C}))) \simeq \pi_1(K) \ .$$

Here, $\pi_1(K)$ is the usual fundamental group from topology.

3.4 Comparing fundamental groups

We have the following issue, or feature.

3.18 Proposition. *Let $\bar{x} \to \mathbf{A}^1_{\mathbf{C}}$ be any geometric point. Then*

$$\left(\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}}, \bar{x})(*)\right)^{\mathrm{ab}} \neq 1 \ .$$

In fact, this group has cardinality at least $2^{\aleph_0}$. Hence

$$\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}}) \neq 1 \ .$$

The same is true for $\mathbf{P}^1_{\mathbf{C}}$.

3.19 Definition.

(1) A condensed set T is *quasicompact* if, for every epimorphism

$$\coprod_{i \in I} S_i \to T \ ,$$

there is a finite subcollection that covers T.

(2) A condensed set T is *quasiseparated* if, for every diagram

$$S \to T \leftarrow S'$$

with S and S' quasicompact, the pullback $S \times_T S'$ is quasicompact.

3.20 Example. If T is a profinite set, then T is qcqs as a condensed set.

3.21 Example. The condensed group $\pi_1^{\mathrm{cond}}(\mathbf{A}^1_{\mathbf{C}})$ is not quasiseparated.

3.22 Theorem (Clausen–Scholze)**.** *The inclusion*

$$\mathrm{Cond}(\mathbf{Set})^{\mathrm{qs}} \subset \mathrm{Cond}(\mathbf{Set})$$

admits a left adjoint $S \mapsto S^{\mathrm{qs}}$ that preserves finite products.

3.23 Warning. The functor $(-)^{\mathrm{qs}}$ does not extend to condensed anima.

3.24 Theorem. *If X is a topologically noetherian scheme that is geometrically unibranch, then the natural map*

$$\big(\pi_1^{\mathrm{cond}}(X)\big)^{\mathrm{qs}} \to \pi_1^{\text{ét}}(X)$$

is an isomorphism.

Fact (Bhatt–Scholze). The group $\pi_1^{\text{proét}}(X)$ is a Noohi group, namely a Hausdorff group G such that

$$G \simeq \mathrm{Aut}(\mathrm{forget}\colon\ \mathbf{Set}^G \to \mathbf{Set})$$

with the compact-open topology.

3.25 Theorem. *If X is a qcqs scheme with finitely many irreducible components, then*

$$\big(\pi_1^{\mathrm{cond}}(X, \bar{x})\big)^{\mathrm{Noohi}} \simeq \pi_1^{\text{proét}}(X, \bar{x})\,.$$

3.5 Relation to logic

Setup. Let T be a complete first-order theory, and let $\mathcal{U}$ be a sufficiently saturated model of T. Lascar defined a quasicompact topological group $\mathrm{Gal}_{\mathrm{L}}(T, \mathcal{U})$. Up to isomorphism, $\mathrm{Gal}_{\mathrm{L}}(T, \mathcal{U})$ does not depend on $\mathcal{U}$. The construction is quite complicated and intricate.

3.26 Notation. Write

$$\mathrm{Mod}_T \coloneqq \mathrm{Mod}_T(\mathbf{Set})$$

for the category of models of T.

3.27 Example. If T is the theory of (abelian) groups, rings, and so on, then Mod_T is the category of (abelian) groups, rings, and so on.

3.28 Theorem (Cousins–Campion–Ye, 2021)**.** *There is an isomorphism of discrete groups*

$$\mathrm{Gal}_{\mathrm{L}}(T, \mathcal{U}) \simeq \pi_1(\mathrm{BMod}_T, \mathcal{U})\,.$$

3.29 Observation. The category Mod_T has a natural enhancement to a condensed category $\mathbf{Mod}_T$ given by

$$K \mapsto \mathrm{Mod}_T(\mathrm{Sh}(K; \mathbf{Set}))\,.$$

3.30 Theorem (Damaj–H.–Zhang)**.** *There is an isomorphism of condensed groups*

$$\mathrm{Gal}_{\mathrm{L}}(T, \mathcal{U}) \simeq \pi_1(\mathrm{B}^{\mathrm{cond}}\mathbf{Mod}_T, \mathcal{U})\,.$$

Relation to the proétale fundamental group

Actually, $\mathbf{Mod}_T$ is a special case of a construction that makes sense for any ∞-topos $\mathcal{X}$. There is a condensed ∞-category

$$\mathbf{Pt}(\mathcal{X}) \in \mathrm{Cond}(\mathbf{Cat}_\infty)$$

given by

$$\mathbf{Pt}(\mathcal{X})(K) := \mathrm{Fun}^*(\mathcal{X}, \mathrm{Sh}(K)) .$$

When $\mathcal{X}$ is the classifying ∞-topos of the theory T, this recovers $\mathbf{Mod}_T$.
Fact. For a qcqs scheme X, the category $\mathrm{Gal}(X)$ is naturally a full condensed subcategory of $\mathbf{Pt}(X_{\text{ét}})$, but $\mathrm{Gal}(X)$ is much smaller.

3.31 Proposition (H. [3])**.** *For each extremally disconnected compact Hausdorff space K, the inclusion*

$$\mathrm{Gal}(X)(K) \subset \mathbf{Pt}(X_{\text{ét}})(K)$$

admits a left adjoint. Hence

$$\mathrm{B}^{\mathrm{cond}}\,\mathrm{Gal}(X) \simeq \mathrm{B}^{\mathrm{cond}}\mathbf{Pt}(X_{\text{ét}}) .$$

Upshot. Up to Noohi completion, $\pi_1^{\text{proét}}(X)$ and $\mathrm{Gal}_{\mathrm{L}}(T)$ are both special cases of

$$\pi_1(\mathrm{B}^{\mathrm{cond}}\mathbf{Pt}(\mathcal{X}))$$

applied to different ∞-topoi.

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